CHAPTER 10

EIGENVALUE METHODS AND BOUNDARY VALUE PROBLEMS

SECTION 10.1

STURM-LIOUVILLE PROBLEMS AND EIGENFUNCTION EXPANSIONS

1. In the notation of Equation (9) in Section 10.1 of the text we have $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$, so Theorem 1 implies that the eigenvalues are all nonnegative. If $\lambda = 0$, then $y'' = 0$ implies that $y(x) = Ax + B$. Then $y'(x) = A$, so the endpoint conditions yield $A = 0$, but B remains arbitrary. Hence $\lambda_0 = 0$ is an eigenvalue with eigenfunction

$$y_0(x) = 1.$$

If $\lambda = \alpha^2 > 0$, then the equation $y'' + \alpha^2 y = 0$ has general solution

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

with

$$y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

Then $y'(0) = 0$ yields $B = 0$ so $A \neq 0$, and then

$$y'(L) = -A\alpha\sin\alpha L = 0,$$

so αL must be an integral multiple of π. Thus the nth positive eigenvalue is

$$\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{L^2},$$

and the associated eigenfunction is

$$y_n(x) = \cos\frac{n\pi x}{L}.$$

2. In the notation of Equation (9) in this section we have $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$, so Theorem 1 implies that the eigenvalues are all nonnegative. If $\lambda = 0$, then $y'' = 0$ implies $y(x) = Ax + B$. But then $y(0) = B = 0$ and $y'(L) = A = 0$, so it follows that 0 is not an eigenvalue. We may therefore write $\lambda = \alpha^2 > 0$, so our equation is

$y'' + \alpha^2 y = 0$ with general solution

$$y(x) = A\cos\alpha x + B\sin\alpha x.$$

Now $y(0) = A = 0$, so $y(x) = B\sin\alpha x$ and

$$y'(x) = B\alpha\cos\alpha x.$$

Hence

$$y'(L) = B\alpha\cos\alpha L = 0,$$

so it follows that αL must be an odd multiple of $\pi/2$. Thus

$$\alpha_n = \frac{(2n-1)\pi}{2L}, \quad \lambda_n = \alpha_n^2, \quad y_n(x) = \sin\alpha_n x.$$

3. If $\lambda = 0$ then $y'' = 0$ yields $y(x) = Ax + B$ as usual. But $y'(0) = A = 0$, and then $hy(L) + y'(L) = h(B) + 0 = 0$, so $B = 0$ also. Thus $\lambda = 0$ is not an eigenvalue. If $\lambda = \alpha^2 > 0$ so our equation is $y'' + \alpha^2 y = 0$, then

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

so

$$y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

Now $y'(0) = 0$ yields $B = 0$, so we may write

$$y(x) = \cos\alpha x, \qquad y'(x) = -\alpha\sin\alpha x.$$

The equation

$$hy(L) + y'(L) = h\cos\alpha L - \alpha\sin\alpha L = 0$$

then gives

$$\tan\alpha L = \frac{h}{\alpha} = \frac{hL}{\alpha L},$$

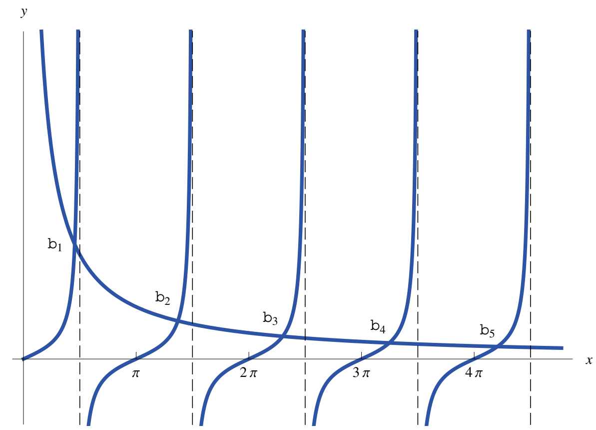
so $\beta_n = \alpha_n L$ is the nth positive root of the equation

$$\tan x = \frac{hL}{x}.$$

Thus

$$\lambda_n = \alpha_n^2 = \frac{\beta_n^2}{L^2}, \qquad y_n(x) = \cos\frac{\beta_n x}{L}.$$

Finally, a sketch of the graphs $y = \tan x$ and $y = hL/x$ indicates that $\beta_n \approx (n-1)\pi$ for n large.



4. Here $\alpha_1 = h > 0$, $\alpha_2 = \beta_1 = 1$, and $\beta_2 = 0$, so by Theorem 1 in Section 10.1 there are no negative eigenvalues. If $\lambda = 0$ and $y(x) = Ax + B$, then the equations

$$hy(0) - y'(0) = hB - A = 0, \qquad y(L) = AL + B = 0$$

imply $h = A/B = -1/L < 0$. Thus 0 is not an eigenvalue. If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

then the condition $hy(0) = y'(0)$ yields $B = hA/\alpha$, so

$$y(x) = \frac{A}{\alpha}(\alpha\cos\alpha x + h\sin\alpha x)$$

$$= \frac{A}{\beta}\left(\beta\cos\frac{\beta x}{L} + hL\sin\frac{\beta x}{L}\right)$$

where $\beta = \alpha L$. Then the condition

$$y(L) = \frac{A}{\beta}(\beta\cos\beta + hL\sin\beta) = 0$$

reduces to $\tan\beta = -\dfrac{\beta}{hL}$.

5. If $\lambda = 0$ then $y'' = 0$ yields $y(x) = Ax + B$ as usual. Then the endpoint condition $hy(0) - y'(0) = 0$ yields $A = hB$. Substitution for A in the other endpoint condition $hy(L) + y'(L) = 0$ then gives $B(h^2L + 2h) = 0$, so it follows that $A = B = 0$. Thus $\lambda = 0$ is not an eigenvalue.

If $\lambda = \alpha^2 > 0$ so our equation is $y'' + \alpha^2 y = 0$, then

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

so

$$y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

Then the endpoint condition $hy(0) - y'(0) = hA - B\alpha = 0$ gives

$$B = \frac{hA}{\alpha}.$$

Substitution of this value of B in the other endpoint condition $hy(L) + y'(L) = 0$ yields an equation that simplifies to

$$(h^2 - \alpha^2)\sin\alpha L + 2h\alpha\cos\alpha L = 0.$$

Substitution of $\beta = \alpha L,\ \alpha = \beta / L$ in this equation then gives the equation

$$(\beta^2 - h^2L^2)\sin\beta = 2hL\beta\cos\beta,$$

whence we see that $\beta_n = \alpha_n L$ is the *n*th positive root of the equation

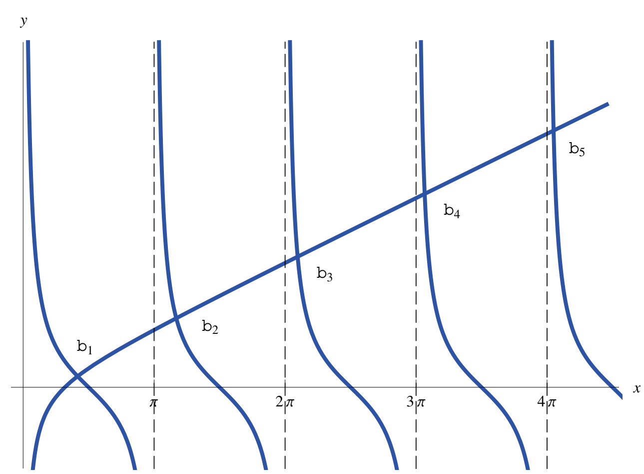
$$\tan x = \frac{2hLx}{x^2 - h^2L^2}, \qquad \text{or} \qquad 2hL\cot x = \frac{x^2 - h^2L^2}{x}.$$

Thus

$$\lambda_n = \alpha_n^2 = \frac{\beta_n^2}{L^2}$$

and

$$y_n(x) = A\cos\alpha_n x + \frac{hLA}{\beta_n}\sin\alpha_n x = \frac{A}{\beta_n}\left(\beta_n\cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right).$$



The figure shows the graphs $y = 2hL\cot x$ and $y = (x^2 - h^2L^2)/x$ (with $hL = 1$ to illustrate the situation). Each intersection point is labeled with its *x*-coordinate β_n. It is apparent that $\beta_n \approx (n-1)\pi$ for n large.

6. $y_n(x) = \sin\dfrac{(2n-1)\pi x}{2L}$ so Equation (25) in Section 10.1 — with $r(x) \equiv 1$ — yields

$$c_n = \frac{\int_0^L f(x)\sin\dfrac{(2n-1)\pi x}{2L}\,dx}{\int_0^L \sin^2\dfrac{(2n-1)\pi x}{2L}\,dx} = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx,$$

because the denominator integral here evaluates — by use of the trigonometric identity $\sin^2 A = \tfrac{1}{2}(1-\cos 2A)$ — to $L/2$.

7. The coefficient c_n in Eq. (23) of this section is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = L$, and $y_n(x) = \sin\dfrac{\beta_n x}{L}$. Using the fact that $\tan\beta_n = -\dfrac{\beta_n}{hL}$, so $\dfrac{\sin\beta_n}{\beta_n} = -\dfrac{\cos\beta_n}{hL}$, we find that

$$\int_0^L \sin^2\frac{\beta_n x}{L}\,dx = \int_0^L \frac{1}{2}\left(1-\cos\frac{2\beta_n x}{L}\right)dx = \frac{1}{2}\left[x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right]_0^L$$

$$= \frac{1}{2}\left(L - L\frac{\sin\beta_n}{\beta_n}\cos\beta_n\right) = \frac{1}{2}\left(L + L\frac{\cos\beta_n}{hL}\cos\beta_n\right) = \frac{hL+\cos^2\beta_n}{2h}$$

and $\displaystyle\int_0^L \sin\frac{\beta_n x}{L}\,dx = \frac{L(1-\cos\beta_n)}{\beta_n}$. Hence the desired eigenfunction expansion is

$$1 = 2hL\sum_{n=1}^{\infty}\frac{1-\cos\beta_n}{\beta_n\left(hL+\cos^2\beta_n\right)}\sin\frac{\beta_n x}{L}.$$

for $0 < x < L$.

8. The coefficient c_n in (23) is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = L$, and $y_n(x) = \cos\beta_n x/L$:

$$c_n = \frac{\int_0^L \cos\dfrac{\beta_n x}{L}\,dx}{\int_0^L \cos^2\dfrac{\beta_n x}{L}\,dx} = \frac{\int_0^L \cos\dfrac{\beta_n x}{L}\,dx}{\int_0^L \dfrac{1}{2}\left(1+\cos\dfrac{2\beta_n x}{L}\right)dx} = \frac{\left[\dfrac{L}{\beta_n}\sin\dfrac{\beta_n x}{L}\right]_0^L}{\left[\dfrac{1}{2}\left(x+\dfrac{L}{2\beta_n}\sin\dfrac{2\beta_n x}{L}\right)\right]}$$

$$= \frac{\dfrac{L}{\beta_n}\sin\beta_n}{\dfrac{1}{2}\left(L + \dfrac{L}{2\beta_n}\sin 2\beta_n\right)} = \frac{4\sin\beta_n}{2\beta_n + \sin 2\beta_n}.$$

Hence the desired eigenfunction expansion is

$$1 = \sum_{n=1}^{\infty} \frac{4\sin\beta_n}{2\beta_n + \sin 2\beta_n}\cos\frac{\beta_n x}{L}.$$

9. The coefficient c_n in (23) is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = 1$, and $y_n(x) = \sin\beta_n x$. Using the fact that $\tan\beta_n = -\beta_n/h$, so $h\sin\beta_n = -\beta_n\cos\beta_n$, we find that

$$\int_0^1 \sin^2\beta_n x\,dx = \int_0^1 \frac{1}{2}(1-\cos 2\beta_n x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2\beta_n x}{2\beta_n}\right]_0^1$$
$$= \frac{1}{2}\left(1 - \frac{\sin\beta_n}{\beta_n}\cos\beta_n\right) = \frac{1}{2}\left(1 + \frac{\cos^2\beta_n}{h}\right) = \frac{h+\cos^2\beta_n}{2h}$$

and

$$\int_0^1 x\sin\beta_n x\,dx = \frac{1}{\beta_n^2}\int_0^1 \beta_n x\sin\beta_n x\cdot\beta_n dx = \frac{1}{\beta_n^2}\int_0^{\beta_n} u\sin u\,du$$
$$= \frac{1}{\beta_n^2}\left[\sin u - u\cos u\right]_0^{\beta_n} = \frac{\sin\beta_n - \beta_n\cos\beta_n}{\beta_n^2}$$
$$= \frac{\sin\beta_n - \beta_n\cos\beta_n}{\beta_n^2} = \frac{(1+h)\sin\beta_n}{\beta_n^2}.$$

It follows that the desired expansion is given by

$$x = 2h(1+h)\sum_{n=1}^{\infty}\frac{\sin\beta_n\sin\beta_n x}{\beta_n^2\left(h+\cos^2\beta_n\right)}$$

for $0 < x < 1$.

10. The coefficient c_n in (23) is given by Formula (25) with $f(x) = x$, $r(x) = 1$, $a = 0$, $b = 1$, and $y_n(x) = \cos\beta_n x$. Integrations similar to those in Problems 8 and 9 give

$$c_n = \frac{\int_0^1 x\cos\beta_n x\,dx}{\int_0^1 \cos^2\beta_n x\,dx} = \frac{4\left(\beta_n\sin\beta_n + \cos\beta_n - 1\right)}{\beta_n\left(2\beta_n + \sin 2\beta_n\right)}.$$

With this value of c_n for $n = 1, 2, 3, \cdots$, the desired eigenfunction expansion is

$$x = \sum_{n=1}^{\infty} c_n \cos \beta_n x.$$

11. If $\lambda = 0$ then $y'' = 0$ implies that $y(x) = Ax + B$. Then $y(0) = 0$ gives $B = 0$, so $y(x) = Ax$. Hence

$$hy(L) - y'(L) = h(AL) - A = A(hL - 1) = 0$$

if and only if $hL = 1$, in which case $\lambda_0 = 0$ has associated eigenfunction $y_0(x) = x$.

12. If $\lambda = -\alpha^2 < 0$, then the general solution of $y'' - \alpha^2 y = 0$ is

$$y(x) = A \cosh \alpha x + B \sinh \alpha x.$$

But then $y(0) = A = 0$, so we may take $y(x) = \sinh \alpha x$. Now the condition $hy(L) = y'(L)$ yields

$$h \sinh \alpha L = \alpha \cosh \alpha L.$$

It follows that $\beta = \alpha L$ must be a root of the equation

$$\tanh x = \frac{x}{hL}.$$

The curve $y = \tanh x$ passes through the origin with slope 1, and is concave upward for $x < 0$, concave downward for $x > 0$. Hence this curve and the straight line $y = x/hL$ intersect other than at the origin if and only if the slope of the line is less than 1 — that is, if and only if $hL > 1$. In this case, with β_0 the positive root of $\tanh x = x/hL$, we have $\lambda_0 = -\beta_0^2$ and $y_0(x) = \sinh \beta_0 x$.

13. If $\lambda = +\alpha^2 > 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A \cos \alpha x + B \sin \alpha x.$$

But then $y(0) = A = 0$, so we may take $y(x) = \sin \alpha x$. Now the condition $hy(L) = y'(L)$ yields

$$h \sin \alpha L = \alpha \cos \alpha L.$$

It follows that $\beta = \alpha L$ must be a root of the equation

$$\tan x = \frac{x}{hL}.$$

So if β_n is the nth positive root of this equation, then $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and the corresponding eigenfunction is $y_n(x) = \sin \beta_n x / L$.

14. With $\lambda = 0$, $y'' = 0$, and hence $y(x) = Ax + B$, we have $y(0) = B = 0$, so $y(x) = Ax$. Then the condition $hy(L) = y'(L)$ reduces to the equation $hL = A$, which is satisfied because $hL = 1$. Thus $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = x$. Together with the positive eigenvalues and associated eigenfunctions provided by Problem 13, this gives the eigenfunction expansion

$$f(x) = c_0 x + \sum_{n=1}^{\infty} c_n \sin \frac{\beta_n x}{L}$$

where $\tan \beta_n = \beta_n$. The coefficients are given by

$$c_0 = \frac{\int_0^L f(x)\, x\, dx}{\int_0^L x^2\, dx} = \frac{3}{L^3} \int_0^L x f(x)\, dx,$$

$$c_0 = \frac{\int_0^L f(x) \sin \beta_n x / L\, dx}{\int_0^L \sin^2 \beta_n x / L\, dx} = \frac{2}{L \sin^2 \beta_n} \int_0^L f(x) \sin \beta_n x / L\, dx,$$

the latter because

$$\int_0^L \sin^2 \beta_n x / L\, dx = \frac{1}{2} \int_0^L \left(1 - \cos 2\beta_n x / L\right) dx = \frac{1}{2} \left[x - \frac{L}{2\beta_n} \sin \frac{2\beta_n x}{L} \right]_0^L$$

$$= \frac{1}{2} \left(L - L \cdot \frac{\sin \beta_n}{\beta_n} \cdot \cos \beta_n \right) = \frac{L}{2} \left(1 - \cos^2 \beta_n\right) = \frac{L \sin^2 \beta_n}{2}.$$

15. If $\lambda_0 = 0$, then a general solution of $y'' = 0$ is $y(x) = Ax + B$. The conditions

$$y(0) + y'(0) = B + A = 0, \qquad y(1) = A + B = 0$$

both say that $B = -A$, so we may take $y_0(x) = x - 1$ as the eigenfunction associated with $\lambda_0 = 0$. If $\lambda = +\alpha^2 < 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A \cos \alpha x + B \sin \alpha x.$$

But $y(0) + y'(0) = A + B\alpha = 0$, so $A = -B\alpha$, and then

$$y(1) = A \cos \alpha + B \sin \alpha = -B\left(\alpha \cos \alpha - \sin \alpha\right) = 0.$$

Thus the possible values of α are the positive roots $\{\beta_n\}$ of the equation $\tan x = x$, and the nth eigenfunction is $y_n(x) = \beta_n \cos \beta_n x - \sin \beta_n x$,

17. The Fourier sine series of the constant function $f(x) \equiv w$ for $0 < x < L$ is

$$w = \frac{4w}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi x}{L}.$$

If $y = \sum b_n \sin n\pi x / L$, then

$$EI\, y^{(4)} = EI \sum_{n=1}^{\infty} \frac{n^4 \pi^4 b_n}{L^4} \sin \frac{n\pi x}{L}.$$

Upon equating coefficients in these two series and solving for b_n, we see that

$$y(x) = \frac{4wL^4}{EI\pi^5} \sum_{n \text{ odd}} \frac{1}{n^5} \sin \frac{n\pi x}{L}.$$

18. By Equation (16) in Section 9.3, the Fourier sine series of $f(x) = bx$ for $0 < x < L$ is

$$bx = \frac{2bL}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}.$$

If $y = \sum b_n \sin n\pi x / L$, then

$$EI\, y^{(4)} = EI \sum_{n=1}^{\infty} \frac{n^4 \pi^4 b_n}{L^4} \sin \frac{n\pi x}{L}.$$

Upon equating coefficients in these two series and solving for b_n, we see that

$$y(x) = \frac{2bL^5}{EI\pi^5} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \sin \frac{n\pi x}{L}.$$

19. With $\lambda = \alpha^4$, the general solution of $y^{(4)} - \alpha^4 y = 0$ is

$$y(x) = A \cosh \alpha x + B \sinh \alpha x + C \cos \alpha x + D \sin \alpha x,$$

and then

$$y'(x) = \alpha(A \sinh \alpha x + B \cosh \alpha x - C \sin \alpha x + D \cos \alpha x).$$

The conditions $y(0) = 0$ and $y'(0) = 0$ yield $C = -A$ and $D = -B$, so now

$$y(x) = A(\cosh \alpha x - \cos \alpha x) + B(\sinh \alpha x - \sin \alpha x).$$

The conditions $y(L) = 0$ and $y'(L) = 0$ yield the two linear equations

$$A(\cosh \alpha L - \cos \alpha L) + B(\sinh \alpha L - \sin \alpha L) = 0,$$

$$A(\sinh \alpha L + \sin \alpha L) + B(\cosh \alpha L - \cos \alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh \alpha L - \cos \alpha L)^2 - (\sinh^2 \alpha L - \sin^2 \alpha L) = 0.$$

Using the facts that $\cosh^2 A - \sinh^2 A = 1$ and $\cos^2 A + \sin^2 A = 1$, this equation simplifies to

$$\cosh \alpha L \cos \alpha L - 1 = 0,$$

so $\beta = \alpha L = x$ satisfies the equation

$$\cosh x \cos x = 1.$$

The eigenvalue corresponding to the nth positive root β_n is

$$\lambda_n = \alpha_n^4 = \left(\frac{\beta_n}{L}\right)^4.$$

Finally the first equation in the pair above yields

$$B = -\frac{\cosh \alpha L - \cos \alpha L}{\sinh \alpha L - \sin \alpha L},$$

so we may take

$$y_n(x) = (\sinh \beta_n - \sin \beta_n)\left(\cosh \frac{\beta_n x}{L} - \cos \frac{\beta_n x}{L}\right) - (\cosh \beta_n - \cos \beta_n)\left(\sinh \frac{\beta_n x}{L} - \sin \frac{\beta_n x}{L}\right)$$

as the eigenfunction associated with the eigenvalue λ_n.

20. As in Problem 19, the solution of $y^{(4)} - \alpha^4 y = 0$ satisfying the left-endpoint conditions $y(0) = 0$ and $y'(0) = 0$ is given by

$$y(x) = A(\cosh \alpha x - \cos \alpha x) + B(\sinh \alpha x - \sin \alpha x).$$

The right-endpoint conditions $y''(L) = 0$ and $y^{(3)}(L) = 0$ now yield the two linear equations

$$A(\cosh \alpha L + \cos \alpha L) + B(\sinh \alpha L + \sin \alpha L) = 0,$$

$$A(\sinh \alpha L - \sin \alpha L) + B(\cosh \alpha L + \cos \alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh \alpha L + \cos \alpha L)^2 - (\sinh^2 \alpha L - \sin^2 \alpha L) = 0.$$

This equation simplifies to

$$\cosh \alpha L \cos \alpha L + 1 = 0,$$

so $\beta = \alpha L = x$ satisfies the equation

$$\cosh x \cos x = -1.$$

The eigenvalue corresponding to the nth root β_n is

$$\lambda_n = \alpha_n^4 = \left(\frac{\beta_n}{L}\right)^4.$$

Finally the first equation in the pair above yields

$$B = -\frac{\cosh \alpha L + \cos \alpha L}{\sinh \alpha L + \sin \alpha L},$$

so we may take

$$y_n(x) = \left(\sinh \beta_n + \sin \beta_n\right)\left(\cosh \frac{\beta_n x}{L} - \cos \frac{\beta_n x}{L}\right)$$
$$-\left(\cosh \beta_n + \cos \beta_n\right)\left(\sinh \frac{\beta_n x}{L} - \sin \frac{\beta_n x}{L}\right)$$

as the eigenfunction associated with the eigenvalue λ_n.

21. As in Problem 19, the solution of $y^{(4)} - \alpha^4 y = 0$ satisfying the left-endpoint conditions $y(0) = 0$ and $y'(0) = 0$ is given by

$$y(x) = A(\cosh \alpha x - \cos \alpha x) + B(\sinh \alpha x - \sin \alpha x).$$

The right-endpoint conditions $y(L) = 0$ and $y''(L) = 0$ yield the two linear equations

$$A(\cosh \alpha L - \cos \alpha L) + B(\sinh \alpha L - \sin \alpha L) = 0,$$
$$A(\cosh \alpha L + \cos \alpha L) + B(\sinh \alpha L + \sin \alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh \alpha L - \cos \alpha L)(\sinh \alpha L + \sin \alpha L) \\ - (\cosh \alpha L + \cos \alpha L)(\sinh \alpha L - \sin \alpha L) = 0.$$

This equation simplifies to $2\cosh\alpha L \sin\alpha L - 2\cos\alpha L \sinh\alpha L = 0$, which is equivalent to $\tanh\alpha L = \tan\alpha L$. Hence $\beta = \alpha L = x$ satisfies the equation $\tanh x = \tan x$, and the eigenvalue corresponding to the nth positive root β_n is $\lambda_n = \alpha_n^4 = (\beta_n / L)^4$.

SECTION 10.2

APPLICATIONS OF EIGENFUNCTION SERIES

1. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 3 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X'(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \cos\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = hL/x$. The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\cos\frac{\beta_n x}{L}.$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx}{\int_0^L \cos^2\frac{\beta_n x}{L}\,dx} = \frac{2h}{hL+\sin^2\beta_n}\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx,$$

because

$$\int_0^L \cos^2\frac{\beta_n x}{L}\,dx = \int_0^L \frac{1}{2}\left(1+\cos\frac{2\beta_n x}{L}\right)dx = \left[\frac{1}{2}\left(x+\frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right)\right]_0^L$$

$$= \frac{1}{2}\left(L+\frac{L}{2\beta_n}\sin 2\beta_n\right) = \frac{1}{2h}\left(hL+\sin\beta_n\cdot\frac{hL\cos\beta_n}{\beta_n}\right)$$

$$= \frac{hL+\sin^2\beta_n}{2h}.$$

In the final step here we use the fact that $(hL\cos\beta_n)/\beta_n = \sin\beta_n$ because $\tan\beta_n = hL/\beta_n$.

2. The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad Y'' - \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Example 5 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2/L^2$ and eigenfunctions

$$X_n(x) = \sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = -x/hL$. The solution of

$$Y_n'' - \frac{\beta_n^2}{L^2}Y_n = 0, \qquad Y(L) = 0$$

is

$$Y_n(y) = \sinh\frac{\beta_n(L-y)}{L},$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{\beta_n x}{L} \sinh\frac{\beta_n (L-y)}{L}.$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\sin\frac{\beta_n x}{L}\,dx}{(\sinh\beta_n)\int_0^L \sin^2\frac{\beta_n x}{L}\,dx} = \frac{4\beta_n}{L(\sinh\beta_n)\left(2\beta_n - \sin 2\beta_n\right)}\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx,$$

because

$$\int_0^L \sin^2\frac{\beta_n x}{L}\,dx = \int_0^L \frac{1}{2}\left(1-\cos\frac{2\beta_n x}{L}\right)dx = \left[\frac{1}{2}\left(x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right)\right]_0^L$$

$$= \frac{1}{2}\left(L - \frac{L}{2\beta_n}\sin 2\beta_n\right) = \frac{L\left(2\beta_n - \sin 2\beta_n\right)}{4\beta_n}.$$

3. The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X'' - \alpha^2 X = 0 \qquad \text{and} \qquad Y'' + \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Problem 3 of Section 10.1 we saw that the Sturm-Liouville problem

$$Y'' + \alpha^2 Y = 0, \qquad Y'(0) = hY(L) + Y'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2/L^2$ and eigenfunctions

$$Y_n(y) = \cos\frac{\beta_n y}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = hL/x$. The solution of

$$X_n'' - \frac{\beta_n^2}{L^2}X_n = 0, \qquad X(L) = 0$$

is

$$X_n(x) = \sinh\frac{\beta_n (L-x)}{L},$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{\beta_n (L-x)}{L}\cos\frac{\beta_n y}{L}.$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L g(y)\cos\frac{\beta_n y}{L}\,dy}{(\sinh\beta_n)\int_0^L \cos^2\frac{\beta_n y}{L}\,dy} = \frac{2h}{(\sinh\beta_n)\left(hL+\sin^2\beta_n\right)}\int_0^L g(y)\cos\frac{\beta_n y}{L}\,dy,$$

because

$$\int_0^L \cos^2\frac{\beta_n y}{L}\,dy = \int_0^L \frac{1}{2}\left(1+\cos\frac{2\beta_n y}{L}\right)dy = \left[\frac{1}{2}\left(y+\frac{L}{2\beta_n}\sin\frac{2\beta_n y}{L}\right)\right]_0^L$$

$$= \frac{1}{2}\left(L+\frac{L}{2\beta_n}\sin 2\beta_n\right) = \frac{hL+\sin^2\beta_n}{2h}.$$

The final step here is the same as in Problem 1, using the fact that $\left(hL\cos\beta_n\right)/\beta_n = \sin\beta_n$ because $\tan\beta_n = hL/\beta_n$.

4. The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X''+\alpha^2 X = 0 \qquad \text{and} \qquad Y''-\alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Example 5 of Section 10.1 we saw that the Sturm-Liouville problem

$$X''+\alpha^2 X = 0, \qquad X(0) = hX(L)+X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2/L^2$ and eigenfunctions

$$X_n(x) = \sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = -x/hL$. The bounded solution of

$$Y_n'' - \frac{\beta_n^2}{L^2}Y_n = 0$$

is

$$Y_n(y) = \exp\left(-\frac{\beta_n y}{L}\right),$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{\beta_n x}{L}\exp\left(-\frac{\beta_n y}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\sin\frac{\beta_n x}{L}\,dx}{\int_0^L \sin^2\frac{\beta_n x}{L}\,dx} = \frac{4\beta_n}{L(2\beta_n - \sin 2\beta_n)}\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx,$$

the calculation of the denominator integral here being the same as in Problem 2.

5. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 4 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad hX(0) - X'(0) = X(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = -x/hL$. The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)dx}{\int_0^L \left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)^2 dx}.$$

The evaluation of the denominator integral here is elementary, but there seems little point in carrying it out explicitly.

6. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 5 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad hX(0) - X'(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation

$$\tan x = \frac{2hLx}{x^2 - h^2L^2}.$$

The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\displaystyle\int_0^L f(x)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)dx}{\displaystyle\int_0^L \left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)^2 dx}.$$

7. The boundary value problem here is

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \quad (0 < x < 1, \quad y > 0) \\ &u_x(0,y) = u(1,y) + u_x(1,t) = 0, \\ &u(x,0) = 100. \end{aligned}$$

The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad Y'' - \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Problem 3 of Section 10.1 we saw (taking $h = L = 1$) that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X'(0) = X(1) + X'(1) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2$ and eigenfunctions

$$X_n(x) = \cos \alpha_n x$$

for $n = 1, 2, 3, \cdots$, with $\{\alpha_n\}$ being the positive roots of the equation $\tan x = 1/x$. The bounded solution of $Y_n'' - \alpha_n^2 Y_n = 0$ is then

$$Y_n(y) = \exp(-\alpha_n y),$$

so the resulting formal series solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n \cos \alpha_n x \exp(-\alpha_n y).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L 100 \cos \alpha_n x \, dx}{\int_0^L \cos^2 \alpha_n x \, dx} = \frac{\left[\dfrac{100}{\alpha_n} \sin \alpha_n x\right]_0^1}{\left[\dfrac{1}{2}\left(x + \dfrac{1}{2\alpha_n} \sin 2\alpha_n x\right)\right]_0^1} = \frac{200 \sin \alpha_n}{\alpha_n + \sin \alpha_n \cos \alpha_n},$$

so

$$u(x, y) = 200 \sum_{n=1}^{\infty} \frac{\sin \alpha_n \cos \alpha_n x \exp(-\alpha_n y)}{\alpha_n + \sin \alpha_n \cos \alpha_n}.$$

The first five positive solutions of $\tan x = 1/x$ are 0.8603, 3.4256, 7.4373, 9.5293, and 12.6453, and we find that

$$u(1,1) \approx 30.8755 + 0.4737 + 0.0074 + 0.0002 + 0.0000 + \cdots \approx 31.4\,^\circ\text{C}.$$

8. With $m = 0$ the boundary value problem in Example 2 is

$$\begin{aligned}
u_{tt} &= a^2 u_{xx} \qquad (0 < x < L, \quad t > 0), \\
u(0,t) &= u_x(L,t) = 0, \\
u_t(x,0) &= 0, \\
u(x,0) &= bx.
\end{aligned}$$

The substitution $u(x,t) = X(x)T(t)$ gives the separated equations

$$X'' + \lambda X = T'' + \lambda a^2 T = 0.$$

and the eigenfunctions of the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(L) = 0$$

are of the form

$$X_n(x) = \sin\frac{n\pi x}{2L}$$

with n odd, with corresponding eigenvalue $\lambda_n = n^2\pi^2/4L^2$. This leads readily to the solution

$$u(x,t) = \sum_{n\text{ odd}} c_n \sin\frac{n\pi x}{2L}\cos\frac{n\pi at}{2L},$$

where c_n is the odd half-multiple sine coefficient (of Problem 21 in Section 9.3) given by

$$c_{2n-1} = \frac{2}{L}\int_0^L bx\sin\frac{(2n-1)\pi x}{L}\,dx = \frac{8bL\sin\dfrac{(2n-1)\pi}{2}}{(2n-1)^2\pi^2} = \frac{8bL(-1)^{n+1}}{(2n-1)^2\pi^2}.$$

9. **(a)** With $\lambda = 0$, the endpoint-value problem in (19) is $X'' = 0$, $X(0) = X'(0) = 0$, which has only the trivial solution $X(x) \equiv 0$. Thus $\lambda = 0$ is not an eigenvalue.

(b) With $\lambda = -\alpha^2 < 0$, the endpoint-value problem in (19) is

$$X'' - \alpha^2 X = 0, \qquad X(0) = 0, \quad -m\alpha^2 X(L) = A\delta X'(L).$$

The differential equation and the left-endpoint condition here give $X(x) = \sinh\alpha x$, and substitution in the right-endpoint condition gives

$$-m\alpha^2\sinh\alpha L = A\delta\alpha\cosh\alpha L, \text{ that is, } \tanh\alpha L = -\frac{k}{\alpha L}$$

with $k = A\delta L/m > 0$. But the graph $y = \tanh x$ lies (aside from the origin) in the first and third quadrants, while the graph $y = -k/x$ lies interior to the second and fourth quadrants. Hence the two cannot intersect, and it follows that there cannot be an eigenvalue of the assumed form $\lambda = -\alpha^2 < 0,$.

10. **(a)** With $\delta = 7.75\ \text{gm/cm}^3$ and $E = 2{\cdot}10^{12}$ in Equation (16), the speed of sound in steel is

$$a = \sqrt{\frac{E}{\delta}} \approx 5.08\ x\ 10^5\ \text{cm/sec} \approx 11364\ \text{mph}.$$

(b) With $\delta = 1\ \text{gm/cm}^3$ and $K = 2.25{\cdot}10^{10}$ in Equation (16), the speed of sound in water is

$$a = \sqrt{\frac{K}{\delta}} \approx 1.50\ x\ 10^5\ \text{cm/sec} \approx 3355\ \text{mph}.$$

11. **(a)** $$a = \sqrt{\frac{K}{\delta}} = \sqrt{\frac{\lambda p}{m/V}} = \sqrt{\frac{\gamma pV}{m}} = \sqrt{\frac{\gamma nRT_K}{nm_0}} = \sqrt{\frac{\gamma RT_K}{m_0}}$$

(b) $$a = \sqrt{\frac{\gamma RT_K}{m_0}} = \sqrt{\frac{1.4\times 8314(273+T_C)}{29}} = \sqrt{\frac{1.4\times 8314\times 273}{29}\left(1+\frac{T_C}{273}\right)}$$

$$\approx 331.02\sqrt{1+\frac{T_C}{273}}\ \frac{\text{m}}{\text{sec}} \approx 740.47\sqrt{1+\frac{T_C}{273}}\ \frac{\text{miles}}{\text{hour}}$$

$$\approx 740.47\left[1+\frac{1}{2}\left(\frac{T_C}{273}\right)+\cdots\right] \approx 740.47 + 1.356T_C$$

12. The boundary value problem is

$$u_{tt} = a^2 u_{xx} \qquad (0 < x < L,\ t > 0)$$
$$u(0, t) = ku(L, t) + AEu_x(L, t) = 0$$
$$u(x, 0) = f(x),$$
$$u_t(x, 0) = 0.$$

Starting with the general solution

$$X(x) = A\cos\alpha x + B\sin\alpha x$$

of $X'' + \alpha^2 X = 0$, the condition $X(0) = 0$ gives $A = 0$, so

$$X(x) = \sin\alpha x, \qquad X'(x) = \alpha\cos\alpha x.$$

Then the condition $kX(L) + AEX'(L) = 0$ yields

$$k \sin \alpha L + AE\alpha \cos \alpha L = 0,$$

which is equivalent to the equation

$$\tan x = -\frac{AEx}{kL}$$

with $x = \alpha L$, $\alpha = x/L$. If $\{\beta_n\}$ are the positive roots of this equation, then the nth eigenvalue is $\lambda_n = \alpha_n^2 = (\beta_n/L)^2$ with associated eigenfunction

$$X_n(x) = \sin \frac{\beta_n x}{L}.$$

The associated function of t is

$$T_n(t) = A_n \cos \frac{\beta_n at}{L} + B_n \sin \frac{\beta_n at}{L},$$

but the condition $T'(0) = 0$ yields $B_n = 0$. Hence we obtain a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{\beta_n x}{L} \cos \frac{\beta_n at}{L}.$$

15. $$\int_0^L \sin \frac{\beta_m x}{L} \sin \frac{\beta_n x}{L} dx = \frac{L}{2}\left[\frac{\sin(\beta_m - \beta_n)}{\beta_m - \beta_n} - \frac{\sin(\beta_m + \beta_n)}{\beta_m + \beta_n}\right]$$

$$= \frac{L}{2\left(\beta_m^2 - \beta_n^2\right)}\left[\begin{array}{l}\left(\beta_m + \beta_n\right)\left(\sin \beta_m \cos \beta_n - \sin \beta_n \cos \beta_m\right) \\ \qquad -\left(\beta_m - \beta_n\right)\left(\sin \beta_m \cos \beta_n + \sin \beta_n \cos \beta_m\right)\end{array}\right]$$

$$= \frac{L}{\beta_m^2 - \beta_n^2}\left[\beta_n \sin \beta_m \cos \beta_n - \beta_m \sin \beta_n \cos \beta_m\right]$$

$$= \frac{L}{\beta_m^2 - \beta_n^2}\left[\beta_n \cdot \frac{M \cos \beta_m}{m\beta_m} \cdot \cos \beta_n - \beta_m \cdot \frac{M \cos \beta_n}{m\beta_n} \cdot \cos \beta_m\right]$$

$$= \frac{LM}{m\left(\beta_m^2 - \beta_n^2\right)} \cos \beta_m \cos \beta_n \left(\frac{\beta_n}{\beta_m} - \frac{\beta_m}{\beta_n}\right) = -\frac{LM \cos \beta_m \cos \beta_n}{m\beta_m \beta_n} \neq 0$$

16. When we substitute $v(r, t) = ru_r(r, t)$ we get the boundary value problem

$$v_t = kv_{rr}$$

$$v(0, \mathrm{t}) = v(a, t) - av_r(a, t) = 0$$

$$v(r,0) = r\,f(r).$$

Then $v(r,t) = R(r)T(t)$ yields the equations

$$R'' + \lambda R = 0, \qquad T' = -\lambda kT.$$

If $\lambda_0 = 0$ then $R(r) = Ar + B$. The condition $R(0) = 0$ gives $B = 0$, and $R(r) = Ar$ satisfies the condition $R(a) - aR'(a) = 0$. Thus $\lambda_0 = 0$ is an eigenvalue with eigenfunction

$$R_0(r) = r; \qquad T_0(t) = 1.$$

If $\lambda = \alpha^2 > 0$ then

$$R(r) = A\cos\alpha r + B\sin\alpha r$$

and $R(0) = 0$ gives $A = 0$, so

$$R(r) = \sin\alpha r, \qquad R'(r) = \alpha\cos\alpha r.$$

The condition $R(a) = aR'(a)$ yields $\sin\alpha a = a\alpha\cos\alpha a$, that is,

$$\tan x = x$$

where $x = \alpha a$. If $\{\beta_n\}$ are the roots of this equation, then $\lambda_n = (\beta_n/a)^2$ is an eigenvalue with associated eigenfunction

$$R_n(r) = \sin\frac{\beta_n r}{a}, \qquad \text{and} \qquad T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{a^2}\right).$$

We therefore obtain a solution of the form

$$v(r,t) = c_0 r + \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{a^2}\right)\sin\frac{\beta_n r}{a}.$$

The coefficient formulas given in the textbook follow immediately from Problem 14 in Section 10.1, and finally we obtain $u(r,t)$ upon division of $v(r,t)$ by r.

18. The only difference from Example 3 in the text is that the solution of Equation (37) with $T_n'(0) = 0$ is $T_n(t) = \sin\dfrac{n^2\pi^2 a^2 t}{L^2}$.

19. With the given initial velocity function $g(x)$ with constant value $P/2\rho\varepsilon$ concentrated in the interval $L/2-\varepsilon < x < L/2+\varepsilon$, the coefficient formula of Problem 18 gives

$$\begin{aligned} c_n &= \frac{2L}{n^2\pi^2a^2}\int_{L/2-\varepsilon}^{L/2+\varepsilon}\frac{P}{2\rho\varepsilon}\sin\frac{n\pi x}{L}\,dx \\ &= \frac{L^2P}{n^3\pi^3a^2\rho\varepsilon}\left[\cos\left(\frac{n\pi}{2}-\frac{n\pi\varepsilon}{L}\right)-\cos\left(\frac{n\pi}{2}+\frac{n\pi\varepsilon}{L}\right)\right] = \frac{2L^2P}{n^3\pi^3a^2\rho\varepsilon}\sin\frac{n\pi}{2}\sin\frac{n\pi\varepsilon}{L}. \end{aligned}$$

This gives the ε-dependent solution

$$y(x,t,\varepsilon) = \frac{2L^2P}{\pi^3a^2\rho\varepsilon}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi\varepsilon}{L}\sin\frac{n^2\pi^2a^2t}{L^2}\sin\frac{n\pi x}{L}.$$

Because

$$\frac{L}{n\pi\varepsilon}\sin\frac{n\pi\varepsilon}{L} = \frac{\sin(n\pi\varepsilon/L)}{n\pi\varepsilon/L} \to 1 \quad \text{as} \quad \varepsilon\to 0,$$

the limit $y(x,t) = \lim_{\varepsilon\to 0} y(x,t,\varepsilon)$ has the expansion

$$y(x,t) = \frac{2LP}{\pi^2a^2\rho}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n^2\pi^2a^2t}{L^2}\sin\frac{n\pi x}{L}.$$

20. $$c_n = \frac{2L}{n^2\pi^2a^2}\int_0^L v_0\sin\frac{n\pi x}{L}\,dx = \frac{2v_0L^2}{n^3\pi^3a^2}(1-\cos n\pi)$$

The fundamental frequency is

$$\omega_1 = \frac{\pi^2a^2}{L^2} = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}\,\omega_1.$$

With

$$\begin{aligned} E &= 2\cdot 10^{12}\ \text{dyne/cm}^2, \\ I &= (2.54)^4/12 \approx 3.47\ \text{cm}^4, \\ \rho &= (7.75)(2.54)^2 \approx 50.00\ \text{gm/cm}, \\ L &= (19)(2.54) \approx 48.26\ \text{cm}, \end{aligned}$$

we calculate

$$\omega_1 \approx 1578\ \text{rad/sec} \approx 251\ \text{cycles/sec}.$$

Thus we hear middle C (approximately).

SECTION 10.3

STEADY PERIODIC SOLUTIONS AND NATURAL FREQUENCIES

In Problems 1–6 we substitute $u(x,t) = X(x)\cos\omega t$ in

$$u_{tt} = a^2 u_{xx} \qquad (a^2 = E/\delta)$$

and then cancel the factor $\cos\omega t$ to obtain the ordinary differential equation

$$a^2X'' + \omega^2 X = 0$$

with general solution

$$X(x) = A\cos\frac{\omega x}{a} + B\sin\frac{\omega x}{a}. \qquad (*)$$

It then remains only to apply the given endpoint conditions to determine the natural (circular) frequencies — the values of ω for which a non-trivial solution exists.

1. Endpoint conditions: $X(0) = X(L) = 0$

The condition $X(0) = 0$ in (*) implies that $A = 0$, so $X(x) = \sin(\omega x/a)$. Then $X(L) = \sin(\omega L/a) = 0$ implies that $\omega L/a = n\pi$, an integral multiple of π. Hence the nth natural frequency is $\omega_n = \dfrac{n\pi a}{L} = \dfrac{n\pi}{L}\sqrt{\dfrac{E}{\delta}}$.

2. Endpoint conditions: $X'(0) = X'(L) = 0$

The condition $X'(0) = 0$ gives $B = 0$ in (*), so we have

$$X(x) = \cos\frac{\omega x}{a}, \qquad \text{so} \qquad X'(x) = -\frac{\omega}{a}\sin\frac{\omega x}{a}.$$

Hence the condition $X'(L) = 0$ implies that $\omega L/a$ is an integral multiple of π. Thus the nth natural frequency is $\omega_n = \dfrac{n\pi a}{L} = \dfrac{n\pi}{L}\sqrt{\dfrac{E}{\delta}}$.

3. Endpoint conditions: $X(0) = X'(L) = 0$

The condition $X(0) = 0$ gives $A = 0$ in (*), so we have

$$X(x) = \sin\frac{\omega x}{a}, \qquad \text{so} \qquad X'(x) = \frac{\omega}{a}\cos\frac{\omega x}{a}.$$

Hence the condition $X'(L) = 0$ implies that $\omega L/a$ is an *odd* integral multiple of $\pi/2$. Thus the *n*th natural frequency is $\omega_n = \dfrac{(2n-1)\pi a}{2L} = \dfrac{(2n-1)\pi}{2L}\sqrt{\dfrac{E}{\delta}}.$

4. Endpoint conditions: $u(0,t) = m u_{tt}(L, t) + AE u_x(L, t) = 0$

The condition $X(0) = 0$ gives $A = 0$ in (*), so we have

$$X(x) = \sin\frac{\omega x}{a}, \qquad \text{so} \qquad u(x,t) = \sin\frac{\omega x}{a}\cos\omega t.$$

Then

$$u_{tt}(x,t) = -\omega^2\sin\frac{\omega x}{a}\cos\omega t, \qquad u_x(x,t) = \frac{\omega}{a}\cos\frac{\omega x}{a}\cos\omega t,$$

so the other endpoint condition is

$$-m\omega^2\sin\frac{\omega L}{a}\cos\omega t + AE\frac{\omega}{a}\cos\frac{\omega L}{a}\cos\omega t = 0.$$

Upon canceling the $\cos\omega t$ factor, we find that

$$\tan\frac{\omega L}{a} = \frac{AE}{ma\omega} = \frac{AEL}{ma^2\cdot\omega L/a} = \frac{AEL}{mE/\rho\cdot\omega L/a} = \frac{AE\rho}{m\cdot\omega L/a} = \frac{M}{m\cdot\omega L/a}.$$

Thus $\beta = \omega L/a$ is a positive root of the equation $\tan x = \dfrac{M/m}{x}$, and the *n*th natural frequency is given by

$$\omega_n = \frac{\beta_n a}{L} = \frac{\beta_n}{L}\sqrt{\frac{E}{\delta}}$$

where β_n is the *n*th positive root of this equation. This is the special case $k = 0$ of Problem 7 below.

5. Endpoint conditions: $u_x(0, t) = ku(L, t) + AE u_x(L, t) = 0$

The condition $X'(0) = 0$ gives $B = 0$ in (*), so we have

$$X(x) = \cos\frac{\omega x}{a}, \qquad \text{so} \qquad u(x,t) = \cos\frac{\omega x}{a}\cos\omega t.$$

Then

$$u_x(x,t) = -\frac{\omega}{a}\sin\frac{\omega x}{a}\cos\omega t,$$

so the other endpoint condition is

$$k\cos\frac{\omega L}{a}\cos\omega t - AE\frac{\omega}{a}\sin\frac{\omega L}{a}\cos\omega t = 0.$$

Upon canceling the $\cos\omega t$ factor, we find that

$$AE\frac{\omega L}{a}\tan\frac{\omega L}{a} = kL.$$

Thus $\beta = \omega L/a$ is a positive root of the equation $AEx\tan x = kL,$ and the nth natural frequency is given by

$$\omega_n = \frac{\beta_n a}{L} = \frac{\beta_n}{L}\sqrt{\frac{E}{\delta}}$$

where β_n is the nth positive root of this equation.

6. Endpoint conditions:

$$m_0u_{tt}(0,t) - AEu_x(0,t) = 0,$$
$$m_1u_{tt}(L,t) + AEu_x(L,t) = 0$$

When we substitute $u(x,t) = X(t)\cos\omega t$ in the two endpoint conditions and then cancel the $\cos\omega t$ factor, we get the equations

$$m_0\omega^2X(0) + KX'(0) = 0$$
$$m_1\omega^2X(L) - KX'(L) = 0$$

where we write $K = AE$ to avoid confusion with the coefficient of $\cos\omega x/a$ in

$$X(x) = A\cos\frac{\omega x}{a} + B\sin\frac{\omega x}{a}.$$

Then

$$X(0) = A, \qquad X'(0) = \frac{B\omega}{a}$$
$$X(L) = A\cos\frac{\omega L}{a} + B\sin\frac{\omega L}{a}$$
$$X'(L) = \frac{\omega}{a}\left(-A\sin\frac{\omega L}{a} + \cos\frac{\omega L}{a}\right).$$

If we write $z = \omega L/a,$ then

$$X(0) = A, \qquad X'(0) = \frac{Bz}{L}$$
$$X(L) = A\cos z + B\sin z$$

$$X'(L) = \frac{z}{L}(-A\sin z + B\cos z).$$

When we substitute these values and $\omega = az/L$ in the two endpoint conditions above and collect coefficients of A and B, we get the equations

$$m_0 a^2 zA + KLB = 0,$$
$$A(m_1 a^2 z\cos z + KL\sin z) + B(m_1 a^2 z\sin z - KL\cos z) = 0.$$

In order for this system to have a non-trivial solution for A and B, its determinant of coefficients must vanish,

$$m_0 a^2 z(m_1 a^2 z\sin z - KL\cos z) - KL(m_1 a^2 z\cos z + KL\sin z) = 0.$$

When we substitute $a^2 = E/\delta$, $M = \delta AL$, and $K = AE$, this last equation simplifies finally to the frequency equation

$$(m_0 m_1 z^2 - M^2)\sin z = M(m_0 + m_1)\, z\cos z.$$

If β_n is the nth positive root, then the nth natural frequency is

$$\omega_n = \frac{\beta_n a}{L} = \frac{\beta_n}{L}\sqrt{\frac{E}{\delta}}.$$

7. Endpoint conditions:

$$u(0,t) = m u_{tt}(L,t) + AE u_x(L,t) + ku(L,t) = 0$$

The condition $u(0,t) = 0$ implies that

$$X(x) = \sin\frac{\omega x}{a}, \qquad \text{so} \qquad X'(x) = \frac{\omega}{a}\cos\frac{\omega x}{a}.$$

When we substitute $u(x,t) = X(x)\cos\omega t$ in the endpoint condition at $x = L$ and cancel the $\cos\omega t$ factor we get

$$-m\omega^2 X(L) + AEX'(L) + kX(L) = 0.$$

Next we substitute

$$z = \omega L/a, \qquad \omega = az/L, \qquad a^2 = E/\delta,$$
$$X(L) = \sin z, \qquad X'(L) = (z/L)\cos z.$$

The result simplifies readily to the frequency equation

$$(mEz^2 - k\delta L^2)\sin z = MEz\cos z.$$

If β_n is the nth positive root, then the nth natural frequency is $\omega_n = \frac{\beta_n a}{L} = \frac{\beta_n}{L}\sqrt{\frac{E}{\delta}}$.

In Problems 8–14 we substitute $y(x, t) = X(x)\cos\omega t$ in

$$y_{tt} + a^4 y_{xxxx} = 0 \qquad (a^4 = EI/\rho)$$

and then cancel the factor $\cos\omega t$ to obtain the ordinary differential equation

$$a^4 X^{(4)} - \omega^2 X = 0$$

with general solution

$$X(x) = A\cosh\frac{\theta x}{a} + B\sinh\frac{\theta x}{a} + C\cos\frac{\theta x}{a} + D\sin\frac{\theta x}{a} \qquad (**)$$

where $\theta = \sqrt{\omega}$. We then get the natural frequencies of vibration by applying the given endpoint conditions.

8. Endpoint conditions: $y(0, t) = y_{xx}(0, t) = 0, \quad y(L, t) = y_{xx}(L, t) = 0$

Just as in Example 3 of Section 10.2, the conditions $X(0) = X''(0) = 0$ imply that $A = C = 0$ in (**), so

$$X(L) = B\sinh\frac{\theta L}{a} + D\sin\frac{\theta L}{a} = 0,$$

$$X''(L) = \frac{\theta^2}{a^2}\left(B\sinh\frac{\theta L}{a} - D\sin\frac{\theta L}{a}\right) = 0.$$

It follows that

$$B\sinh\frac{\theta L}{a} = D\sin\frac{\theta L}{a} = 0.$$

But $\sinh\theta L/a \neq 0$ so $B = 0$. Hence $D \neq 0$ so $\sin\theta L/a = 0$. Thus $\theta L/a = n\pi$, an integral multiple of π. Therefore the nth natural frequency $\omega_n = \theta_n^2$ is given by

$$\omega_n = \frac{n^2\pi^2 a^2}{L^2} = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

9. Endpoint conditions: $y(0, t) = y_x(0, t) = 0, \quad y(L, t) = y_{xx}(L, t) = 0$

Just as in Problem 21 of Section 10.1, the endpoint conditions $X(0) = X'(0) = 0$ and

$X(L) = X''(L) = 0$ imply that

$$\lambda_n = \frac{\omega_n^2}{a^4} = \left(\frac{\beta_n}{L}\right)^4$$

where β_n is the nth positive zero of the frequency equation

$$\tanh x = \tan x.$$

Therefore the nth natural frequency ω_n is given by

$$\omega_n = \left(\frac{\beta_n}{L}\right)^2 a^2 = \frac{\beta_n^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

10. Endpoint conditions: $y(0, t) = y_x(0, t) = 0, \quad y_{xx}(L, t) = y_{xxx}(L, t) = 0$

Here we have the equation

$$X^{(4)} - \lambda X = 0$$

with $\lambda = \omega^2/a^4$ and endpoint conditions

$$X(0) = X'(0) = X''(L) = X^{(3)}(L) = 0.$$

According to Problem 20 in Section 10.1 the nth eigenvalue is

$$\lambda_n = \left(\frac{\omega_n}{a^2}\right)^2 = \left(\frac{\beta_n}{L}\right)^4$$

where the $\{\beta_n\}$ are the positive roots of the equation

$$\cosh x \cos x = -1.$$

Thus the nth natural frequency is

$$\omega_n = a^2\sqrt{\lambda_n} = \left(\frac{\beta_n}{L}\right)^2 a^2 = \left(\frac{\beta_n}{L}\right)^2\sqrt{\frac{EI}{\rho}}.$$

11. Endpoint conditions: $y(0, t) = y_x(0, t) = 0, \quad y_x(L, t) = y_{xxx}(L, t) = 0$

Here we have the equation

$$X^{(4)} - \lambda X = 0$$

with $\lambda = \omega^2/a^4 = \theta^4/a^4 = \alpha^4$ and endpoint conditions

$$X(0) = X'(0) = X'(L) = X^{(3)}(L) = 0.$$

The left-endpoint conditions readily give $C = -A$ and $D = -B$ in (**), so

$$\begin{aligned} X(x) &= A\cosh\alpha x + B\sinh\alpha x - A\cos\alpha x - B\sin\alpha x. \\ &= A(\cosh\alpha x - \cos\alpha x) + B(\sinh\alpha x - \sin\alpha x). \end{aligned}$$

Then the right-endpoint conditions give

$$\begin{aligned} A(\sinh\alpha L + \sin\alpha L) + B(\cosh\alpha L - \cos\alpha L) &= 0, \\ A(\sinh\alpha L - \sin\alpha L) + B(\cosh\alpha L + \cos\alpha L) &= 0. \end{aligned}$$

The determinant of coefficients of A and B must vanish if there is to be a nontrivial solution, so

$$\begin{aligned} &(\sinh\alpha L + \sin\alpha L)(\cosh\alpha L + \cos\alpha L) \\ &\quad -(\sinh\alpha L - \sin\alpha L)(\cosh\alpha L - \cos\alpha L) = 0. \end{aligned}$$

This equation simplifies to $2\sinh\alpha L\cos\alpha L + 2\cosh\alpha L\sin\alpha L = 0,$ which upon division by $\cosh\alpha L\cos\alpha L$ gives the frequency equation

$$\tanh x + \tan x = 0$$

for $\beta = \alpha L.$ Then the nth frequency is given as usual by

$$\omega_n = \alpha_n^2 a^2 = \frac{\beta_n^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

12. This problem is the special case $k = 0$ of Problem 14 below.

13. This problem is the special case $m = 0$ of Problem 14 below.

14. Endpoint conditions:

$$y(0, t) = y_x(0, t) = y_{xxx}(L, t) = 0$$

$$my_{tt}(L, t) = EIy_{xxx}(L, t) - ky(L, t)$$

With $p = \theta/a,\ \theta = \sqrt{\omega}$ we may write

$$X(x) = A\cosh px + B\sinh px + C\cos px + D\sin px.$$

The conditions $X(0) = X'(0) = 0$ readily imply that $C = -A$ and $D = -B$, so

$$\begin{aligned}
X &= A(\cosh px - \cos px) + B(\sinh px - \sin px), \\
X' &= pA(\sinh px + \sin px) + pB(\cosh px - \cos px), \\
X'' &= p^2A(\cosh px + \cos px) + p^2B(\sinh px + \sin px), \\
X^{(3)} &= p^3A(\sinh px - \sin px) + p^3B(\cosh px + \cos px).
\end{aligned}$$

The endpoint conditions at $x = L$ are

$$\begin{aligned}
&X''(L) = 0, \\
&(k - m\omega^2)X(L) - EIX^{(3)}(L) = 0.
\end{aligned}$$

When we substitute the derivatives above and write $z = pL$ we get

$$A\,(\cosh z + \cos z) + B\,(\sinh z + \sin z) = 0,$$

$$\begin{aligned}
&A\,[(k - m\omega^2)(\cosh z - \cos z) - EIp^3(\sinh z - \sin z)] \\
&+ B\,[(k - m\omega^2)(\sinh z - \sin z) - EIp^3(\cosh z + \cos z)] = 0.
\end{aligned}$$

If Δ denotes the coefficient determinant of these two linear equations in A and B, then the necessary condition $\Delta = 0$ for a non-trivial solution reduces eventually to the equation

$$EIp^3(1 + \cosh z \cos z) - (k - m\omega^2)(\sinh z \cos z - \cosh z \sin z) = 0.$$

Finally we substitute $p = z/L$, $M = \rho L$, and

$$\omega^2 = p^4 a^4 = (z^4/L^4)(EI/\rho)$$

to get the frequency equation

$$MEIz^3(1 + \cosh z \cos z) = (kML^3 - mEIz^4)(\sinh z \cos z - \cosh z \sin z).$$

We may divide by $\cosh z \cos z$ to write this equation in the form

$$MEIz^3(1 + \operatorname{sech} z \sec z) = (kML^3 - mEIz^4)(\tanh z - \tan z).$$

If β_n denotes the nth positive root of this equation, then as usual the nth natural frequency is

$$\omega_n = \frac{\beta_n^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

15. We want to calculate the fundamental frequency of transverse vibration of a cantilever with the numerical parameters

$$L = 400 \text{ cm}$$
$$E = 2 \cdot 10^{12} \text{ gm/cm-sec}^2$$
$$I = (1/12)(30 \text{ cm})(2 \text{ cm})^3 = 20 \text{ cm}^4$$
$$\rho = (7.75 \text{ gm/cm}^3)(60 \text{ cm}^2) = 465 \text{ gm/cm.}$$

When we substitute these values and $\beta_1 = 1.8751$ in the frequency formula

$$\omega_1 = \frac{\beta_1^2}{L^2}\sqrt{\frac{EI}{\rho}},$$

we find that $\omega_1 \approx 6.45$ rad/sec, so the fundamental frequency is $\omega_1/2\pi \approx 1.03$ cycles/sec. Thus the diver should bounce up and down on the end of the diving board about once every second.

16. When we substitute $y(x, t) = X(x)\cos \omega t$ in the given partial differential equation

$$\rho\frac{\partial^2 y}{\partial t^2} + P\frac{\partial^2 y}{\partial x^2} + EI\frac{\partial^4 y}{\partial x^4} = 0$$

and cancel the factor $\cos \omega t$, we get the ordinary differential equation

$$EIX^{(4)} + PX'' - \lambda X = 0$$

where $\lambda = \rho\omega^2$. By solving the characteristic equation

$$EI\, r^4 + Pr^2 - \lambda = EI(r^2 - \alpha^2)(r^2 + \beta^2) = 0$$

we find the general solution

$$X(x) = A \cosh \alpha x + B \sinh \alpha x + C \cos \beta x + D \sin \beta x$$

where

$$\alpha^2 = \frac{-P + \sqrt{P^2 + 4\lambda EI}}{2EI}, \qquad \beta^2 = -\frac{-P - \sqrt{P^2 + 4\lambda EI}}{2EI}.$$

The endpoint conditions $X(0) = X''(0) = 0$ imply that $A = C = 0$, so

$$X(x) = B \sinh \alpha x + D \sin \beta x.$$

Then the conditions $X(L) = X''(L) = 0$ yield the equations

$$B \sinh \alpha L + D \sin \beta L = 0,$$
$$\alpha^2 B \sinh \alpha L - \beta^2 D \sin \beta L = 0.$$

The determinant of these two linear equations in B and D must vanish in order that a nontrivial solution exist, so

$$(\alpha^2 + \beta^2)\sinh \alpha L \sin \beta L = 0.$$

It follows that $\sin \beta L = 0$, so βL must be an integral multiple of π. The definitions of α^2 and β^2 imply that

$$\beta^2 - \alpha^2 = \frac{P}{EI}, \qquad \alpha^2\beta^2 = \frac{\lambda}{EI}.$$

Hence if $\beta_n = n\pi/L$, the corresponding value of α_n is

$$\alpha_n = \sqrt{\frac{n^2\pi^2}{L^2} - \frac{P}{EI}}.$$

Then the corresponding value of λ is

$$\lambda_n = EI\alpha_n^2\beta_n^2 = EI\left(\frac{n^4\pi^4}{L^4}\right)\left(1 - \frac{PL^2}{n^2\pi^2 EI}\right).$$

Finally, the nth natural frequency is given by

$$\omega_n = \sqrt{\frac{\lambda_n}{\rho}} = \frac{n^2\pi^2}{L^2}\left(1 - \frac{PL^2}{n^2\pi^2 EI}\right)^{1/2}\sqrt{\frac{EI}{\rho}}.$$

17. When we substitute $y(x, t) = X(x)\cos \omega t$ in the given partial differential equation

$$\rho\frac{\partial^2 y}{\partial t^2} - \frac{I}{A}\frac{\partial^4 y}{\partial x^2 \partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = 0$$

and cancel the factor $\cos \omega t$, we get the ordinary differential equation

$$EIX^{(4)} + PX'' - \lambda X = 0$$

where $P = \lambda I / \rho A$ and $\lambda = \rho\omega^2$. By solving the characteristic equation

$$EI\, r^4 + Pr^2 - \lambda = EI(r^2 - \alpha^2)(r^2 + \beta^2) = 0$$

we find the general solution

$$X(x) = A\cosh \alpha x + B\sinh \alpha x + C\cos \beta x + D\sin \beta x$$

where

$$\alpha^2 = \frac{-P+\sqrt{P^2+4\lambda EI}}{2EI}, \qquad \beta^2 = -\frac{-P-\sqrt{P^2+4\lambda EI}}{2EI}.$$

The endpoint conditions $X(0) = X''(0) = 0$ imply that $A = C = 0$, so

$$X(x) = B \sinh \alpha x + D \sin \beta x.$$

Then the conditions $X(L) = X''(L) = 0$ yield the equations

$$B \sinh \alpha L + \quad D \sin \beta L = 0,$$
$$\alpha^2 B \sinh \alpha L - \beta^2 D \sin \beta L = 0.$$

The determinant of these two linear equations in B and D must vanish in order that a nontrivial solution exist, so

$$(\alpha^2 + \beta^2)\sinh \alpha L \sin \beta L = 0.$$

It follows that $\sin \beta L = 0$, so βL must be an integral multiple of π. The definitions of α^2 and β^2 imply that

$$\beta^2 - \alpha^2 = \frac{P}{EI} = \frac{\lambda}{\rho AE}, \qquad \alpha^2\beta^2 = \frac{\lambda}{EI}.$$

Hence if $\beta_n = n\pi/L$, the corresponding value of α_n is given by

$$\alpha_n^2 = \frac{n^2\pi^2}{L^2} - \frac{\lambda_n}{\rho AE}.$$

Then $\alpha_n^2\beta_n^2 = \lambda_n / EI$ gives the equation

$$\left(\frac{n^2\pi^2}{L^2} - \frac{\lambda_n}{\rho AE}\right)\frac{n^2\pi^2}{L^2} = \frac{\lambda_n}{EI}$$

that we readily solve for λ_n. The resulting value of the nth natural frequency is

$$\omega_n = \sqrt{\frac{\lambda_n}{\rho}} = \frac{n^2\pi^2}{L^2}\left(1 + \frac{n^2\pi^2 I}{\rho AL^2}\right)^{-1/2}\sqrt{\frac{EI}{\rho}}.$$

18. Substitution of $u(x,t) = X(x)\sin \omega t$ in the longitudinal bar problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad \left(a^2 = \frac{E}{\rho}\right)$$

$$u(0,t) = 0, \quad AEu_x(L,t) = F_0 \sin \omega t$$

yields the endpoint problem

$$X'' + \frac{\omega^2}{a^2} X = 0, \quad X(0) = 0, \quad AE\,X'(L) = F_0.$$

Because of the right-endpoint condition, we try $X(x) = B \sin \omega x / a$ and get

$$AE \cdot B \cdot \frac{\omega}{a} \cos \frac{\omega L}{a} = F_0, \quad \text{so} \quad B = \frac{F_0 a}{AE\omega \cos(\omega L / A)}.$$

The resulting steady periodic solution is

$$u(x,t) = \frac{F_0 a \sin(\omega x / a) \sin \omega t}{AE\omega \cos(\omega L / A)}.$$

19. Substitution of $y(x,t) = X(x) \sin \omega t$ in the transverse bar problem

$$\frac{\partial^2 y}{\partial t^2} + a^4 \frac{\partial^4 y}{\partial x^4} = 0 \qquad \left(a^4 = \frac{EI}{\rho}\right)$$

$$y(0,t) = y_x(0,t) = 0,$$

$$y_{xx}(L,t) = EI\, y_{xxx}(L,t) + F_0 \sin \omega t = 0$$

yields the endpoint problem

$$X^{(4)} - p^4 X = 0 \qquad (\text{where } p^2 = \omega / a^2),$$

$$X(0) = X'(0) = 0,$$

$$X''(L) = EI\, X'''(L) + F_0 = 0.$$

When we impose the fixed-end conditions $X(0) = X'(0) = 0$ on the general solution

$$X(x) = A \cosh px + B \sinh px + C \cos px + D \sin px$$

we find readily that $C = -A$ and $D = -B$, so

$$X(x) = A(\cosh px - \cos px) + B(\sinh px - \sin px).$$

It remains only to find A and B. But the free-end conditions yield the linear equations

$$A(\cosh pL + \cos pL) + B(\sinh pL + \sin pL) = 0$$
$$A(\sinh pL - \sin pL) + B(\cosh pL + \cos pL) = -F_0/p^3EI$$

that can be solved for

$$A = K(\sinh pL + \sin pL), \qquad B = -K(\cosh pL + \cos pL)$$

where

$$K = \frac{F_0}{2EIp^3(1 + \cosh pL \cos pL)}.$$

22. When we substitute

$$e(x,t) = E(x)e^{i\omega t}$$

in the partial differential equation

$$e_{xx} = LCe_{tt} + (LG + RC)e_t + RGe$$

and cancel the factor $e^{i\omega t}$, the result is the ordinary differential equation

$$E''(x) - \gamma E(x) = 0$$

where

$$\gamma = (RG - LC\omega^2) + i\,\omega(LG + RC).$$

If $(\alpha + \beta i)^2 = \gamma$, then the general solution is

$$E(x) = Ae^{-\alpha x}e^{-i\beta x} + Be^{\alpha x}e^{i\beta x}.$$

In order that $e(x,t)$ be bounded as $x \to \infty$ we choose $B = 0$, and in order that $e(0,t) = E_0 \cos \omega t$ we choose $A = E_0$. Then our steady periodic solution is the real part

$$\text{Re}[E(x)e^{i\omega t}] = \text{Re}[E_0e^{-\alpha x}e^{-i\beta x}e^{i\omega t}] = E_0e^{-\alpha x}\cos(\omega t - \beta x).$$

SECTION 10.4

CYLINDRICAL COORDINATE PROBLEMS

1. Substitution of $u(r,t) = R(r)T(t)$ in the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)$$

yields the separation

$$\frac{T''}{a^2 T} = \frac{R''+\frac{1}{r}R'}{R} = \lambda = -\alpha^2.$$

The t-equation has general solution

$$T(t) = A\cos\alpha at + B\sin\alpha at,$$

and we choose $B = 0$, so that $T'(0) = 0$ (because the membrane is initially at rest).
The r-equation can be written in the form

$$r^2R''+rR'+\alpha^2r^2R = 0,$$

which is the parametric Bessel equation of order zero, with continuous solution $R(r) = J_0(\alpha r)$. In order that the fixed boundary condition $R(c) = 0$ be satisfied, we choose $\alpha = \gamma_n / c$, where γ_n is the nth positive solution of $J_0(x) = 0$. At this point we have product functions of the form $J_0(\gamma_n r / c)\cos(\gamma_n at / c)$ that satisfy the wave equation and the homogeneous boundary conditions, so we form the formal series solution

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\cos\frac{\gamma_n at}{c}.$$

In order to satisfy the initial position condition $u(r,0) = f(x)$ is suffices that the $\{c_n\}$ be the Fourier-Bessel coefficients of the function $f(x)$ given by

$$c_n = \frac{2}{c^2\left[J_1(\gamma_n)\right]^2}\int_0^c rf(r)J_0\left(\frac{\gamma_n r}{c}\right)dr.$$

2. This is the same as Problem 1, except that the membrane has initial position $u(r,0) = 0$, so in the t-factor $T(t) = A\cos\alpha at + B\sin\alpha at$ we choose $A = 0$ so that $T(0) = 0$. We then get product functions of the form $J_0(\gamma_n r / c)\sin(\gamma_n at / c)$ that satisfy the wave equation and the homogeneous boundary conditions, so we form the formal series solution

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\sin\frac{\gamma_n at}{c}.$$

In order that the initial velocity condition $u_t(r,0) \equiv v_0$ we satisfied, we want

$$v_0 = \sum_{n=1}^{\infty} \frac{\gamma_n a}{c} \cdot c_n J_0\left(\frac{\gamma_n r}{c}\right),$$

and hence

$$c_n = \frac{c}{\gamma_n a} \cdot \frac{2}{c^2 J_1(\gamma_n)^2} \int_0^c r\, v_0 J_0\left(\frac{\gamma_n r}{c}\right) dr$$

$$= \frac{2cv_0}{a\gamma_n^3 J_1(\gamma_n)^2} \int_0^{\gamma_n} x J_0(x)\, dx \qquad (\text{with } x = \gamma_n r / c)$$

$$= \frac{2cv_0}{a\gamma_n^3 J_1(\gamma_n)^2} \left[x J_1(x) \right]_0^{\gamma_n} = \frac{2cv_0}{a\gamma_n^2 J_1(\gamma_n)}.$$

This gives the desired solution

$$u(r,t) = \frac{2cv_0}{a} \sum_{n=1}^{\infty} \frac{J_0(\gamma_n r / c) \sin(\gamma_n a t / c)}{\gamma_n^2 J_1(\gamma_n)}.$$

3. **(a)** As in Problem 2,

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right) \sin \frac{\gamma_n a t}{c}.$$

In order to satisfy the given initial condition we must choose

$$c_n = \frac{c}{\gamma_n a} \cdot \frac{2}{c^2 J_1(\gamma_n)^2} \int_0^{\varepsilon} \left(\frac{P_0}{\rho \pi \varepsilon^2}\right) r J_0\left(\frac{\gamma_n r}{c}\right) dr$$

$$= \frac{2P_0 c}{\rho \pi \varepsilon^2 \gamma_n^3 a J_1(\gamma_n)^2} \int_0^{\gamma_n \varepsilon / c} x J_0(x)\, dx \qquad (\text{with } x = \gamma_n r / c)$$

$$= \frac{2P_0 c}{\rho \pi \varepsilon^2 \gamma_n^3 a J_1(\gamma_n)^2} \cdot \frac{\gamma_n \varepsilon}{c} J_1\left(\frac{\gamma_n \varepsilon}{c}\right).$$

$$c_n = \frac{2aP_0}{\pi c \rho a^2 \gamma_n J_1(\gamma_n)^2} \cdot \frac{J_1(\gamma_n \varepsilon / c)}{\gamma_n \varepsilon / c}.$$

(b) The final formula given in the text for $u(r, t)$ now follows because $\rho a^2 = T$ and $J_1(x)/x \to 1/2$ as $x \to 0$.

4. **(a)** Just as in Example 1 of the text we derive first a formal series solution of the form

$$u(r,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\gamma_n^2 k t}{c^2}\right) J_0\left(\frac{\gamma_n r}{c}\right).$$

In order to satisfy the given initial condition we calculate the Fourier-Bessel coefficient

$$\begin{aligned}
c_n &= \frac{2}{c^2 J_1(\gamma_n)^2}\int_0^{\varepsilon}\left(\frac{q_0}{s\pi\varepsilon^2}\right) r\,J_0\left(\frac{\gamma_n r}{c}\right) dr \\
&= \frac{2q_0}{s\pi\varepsilon^2\gamma_n^2 J_1(\gamma_n)^2}\int_0^{\gamma_n\varepsilon/c} x\,J_0(x)\,dx \qquad (\text{with } x = \gamma_n r/c) \\
&= \frac{2q_0}{s\pi\varepsilon^2\gamma_n^2 J_1(\gamma_n)^2}\cdot\frac{\gamma_n\varepsilon}{c}\,J_1\left(\frac{\gamma_n\varepsilon}{c}\right). \\
c_n &= \frac{2q_0}{s\pi c^2 J_1(\gamma_n)^2}\cdot\frac{J_1(\gamma_n\varepsilon/c)}{\gamma_n\varepsilon/c}.
\end{aligned}$$

(b) The final formula given in the text for $u(r, t)$ now follows because $J_1(x)/x \to 1/2$ as $x \to 0$.

5. **(a)** We start with the steady-state boundary value problem

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} &= 0 \qquad (r < c, \quad 0 < z < L) \\
u(c, z) &= 0 \\
u(r, 0) &= 0, \\
u(r, L) &= u_0.
\end{aligned}$$

The substitution $u(r, z) = R(r)Z(z)$ yields the equations

$$rR'' + R' + \alpha^2 rR = 0, \qquad Z'' - \alpha^2 Z = 0$$

with separation constant $\lambda = \alpha^2$. The homogeneous endpoint conditions are

$$R(c) = Z(0) = 0.$$

If $\lambda = \alpha^2 = 0$ then $rR'' + R = 0$ implies

$$R(r) = A + B\ln r.$$

We choose $B = 0$ for continuity at $r = 0$, so $R(r) = A$. Then $R(c) = 0$, so $A = 0$ also, and hence 0 is not an eigenvalue.

If $\lambda = \alpha^2 > 0$ then we have the parametric Bessel equation with general solution

$$R(r) = AJ_0(\alpha r) + B\,Y(\alpha r).$$

In order that $R(r)$ be continuous at $r = 0$ we choose $B = 0$, so $R(r) = AJ_0(\alpha r)$. Then

$$R(c) = \alpha A J_0(\alpha c) = 0$$

requires that $\gamma = \alpha c$ be a root of the equation

$$J_0(x) = 0.$$

If $\alpha_n = \gamma_n / c$ where γ_n is the nth positive root of this equation, then

$$R_n(r) = J_0\left(\frac{\gamma_n r}{c}\right).$$

The corresponding function $Z(z)$ of z is

$$Z_n(z) = A_n \cosh\frac{\gamma_n z}{c} + B_n \sinh\frac{\gamma_n z}{c},$$

and we choose $A_n = 0$ because $Z(0) = 0$. Thus we get the formal series solution

$$u(r,z) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\sinh\frac{\gamma_n z}{c}$$

where $J_0(\gamma_n) = 0$. To satisfy the condition $u(r, L) = u_0$, we need (by Eq. (22) in the text)

$$\begin{aligned}
c_n &= \frac{1}{\sinh(\gamma_n L/c)} \cdot \frac{2}{c^2 J_1(\gamma_n)^2} \int_0^c r\, u_0 J_0\left(\frac{\gamma_n r}{c}\right) dr \\
&= \frac{2u_0}{\gamma_n^2 J_1(\gamma_n)^2 \sinh(\gamma_n L/c)} \int_0^{\gamma_n} x J_0(x)\, dx \qquad (\text{with } x = \gamma_n r/c) \\
&= \frac{2u_0}{\gamma_n^2 J_1(\gamma_n)^2 \sinh(\gamma_n L/c)} \Big[x J_1(x)\Big]_0^{\gamma_n} = \frac{2u_0}{\gamma_n J_1(\gamma_n)}.
\end{aligned}$$

This gives the desired solution

$$u(r,t) = 2u_0 \sum_{n=1}^{\infty} \frac{J_0(\gamma_n r/c)\sinh(\gamma_n z/c)}{\gamma_n J_1(\gamma_n)\sinh(\gamma_n L/c)}.$$

6. **(a)** Here we have the same separation of variables

$$rR'' + R' + \alpha^2 rR = 0, \qquad Z'' - \alpha^2 Z = 0$$

as in Problem 5, but the insulation condition $u_r(c,z) = 0$ implies

If $\lambda = \alpha^2 = 0$ then $rR'' + R = 0$ implies

$$R(r) = A + B \ln r.$$

We choose $B = 0$ for continuity at $r = 0$, so $R(r) = A$. Then $R'(c) = 0$, so $\lambda_0 = 0$ is an eigenvalue, and we may take $R_0(r) = 1$. The equation $Z''(z) = 0$ implies $Z(z) = Az + B$, but $Z(0) = 0$ implies $B = 0$, so we take $Z_0(z) = z$.

If $\lambda = \alpha^2 > 0$ then we have the parametric Bessel equation with continuous solution $R(r) = AJ_0(\alpha r)$. Then

$$R'(c) = \alpha A J_0'(\alpha c) = 0$$

requires that $\gamma = \alpha c$ be a root of the equation

$$J_0'(x) = 0.$$

If $\alpha_n = \gamma_n/c$ where γ_n is the nth positive root of this equation, then

$$R_n(r) = J_0\left(\frac{\gamma_n r}{c}\right).$$

The corresponding function $Z(z)$ of z is

$$Z_n(z) = A_n \cosh\frac{\gamma_n z}{c} + B_n \sinh\frac{\gamma_n z}{c},$$

and we choose $A_n = 0$ because $Z(0) = 0$. Thus we get the solution

$$u(r,z) = c_0 z + \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\sinh\frac{\gamma_n z}{c}$$

where $J_0'(\gamma_n) = 0$. To satisfy the condition $u(r, L) = f(r)$ we apply the formulas in (24) in the text and choose

$$c_0 = \frac{2}{Lc^2}\int_0^c r\,f(r)\,dr,$$

$$c_n = \frac{2}{c^2 \sinh(\gamma_n L/c) J_0(\gamma_n)^2}\int_0^c r\,f(r) J_0\left(\frac{\gamma_n r}{c}\right)dr.$$

(b) If $f(r) = u_0$ (constant), then the coefficient formulas above readily yield $c_0 = u_0/L$ and $c_n = 0$ for $n > 0$, the latter because

$$\int x J_0(x)\,dx = xJ_1(x)+C = -xJ_0'(x)+C.$$

Hence the series reduces to the solution $u(r, z) = u_0 z/L$ that one might well guess without all these computations.

7. We want to solve the boundary value problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2} = 0 \qquad (r < 1,\ z > 0)$$

$$hu(1, z) + u_r(1, z) = 0$$

$$u(r, z) \text{ bounded as } z \to \infty$$

$$u(r, 0) = u_0.$$

We start with the separation of variables in Problem 5,

$$rR'' + R' + \alpha^2 rR = 0, \qquad Z'' - \alpha^2 Z = 0$$

and readily see that $\alpha = 0$ is not an eigenvalue. When we impose the condition

$$hR(1) + R'(1) = 0$$

on $R(r) = J_0(\alpha r)$, we find that α must satisfy the equation

$$hJ_0(x) + xJ_0'(x) = 0$$

that corresponds to Case 2 with $n = 0$ in Figure 10.4.2 of the text. If $\{\gamma_n\}$ are the positive roots of this equation then

$$R_n(r) = J_0(\gamma_n r).$$

The general solution of $Z'' = \gamma_n^2 Z$ is

$$Z_n(z) = A_n\exp(-\gamma_n z) + B_n\exp(\gamma_n z),$$

and we choose $B_n = 0$ so that $Z_n(z)$ will be bounded as $z \to \infty$. Thus we obtain a solution of the form

$$u(r,z) = \sum_{n=1}^{\infty} c_n \exp(-\gamma_n z) J_0(\gamma_n r)$$

where

$$hJ_0(\gamma_n) + \gamma_n J_0'(\gamma_n) = 0,$$

so $\gamma_n J_1(\gamma_n) = h J_0(\gamma_n)$ because $J_0' = -J_1$. Finally, Eq. (23) in the text gives

$$\begin{aligned} c_n &= \frac{2\gamma_n^2}{c^2\left(\gamma_n^2 + h^2\right) J_0(\gamma_n)^2} \int_0^c r u_0 J_0\left(\frac{\gamma_n r}{c}\right) dr \\ &= \frac{2u_0}{\left(\gamma_n^2 + h^2\right) J_0(\gamma_n)^2} \int_0^{\gamma_n} x J_0(x)\, dx \qquad \left(\text{with } x = \gamma_n r / c\right) \\ &= \frac{2u_0}{\left(\gamma_n^2 + h^2\right) J_0(\gamma_n)^2} \left[x J_1(x)\right]_0^{\gamma_n} \\ &= \frac{2u_0 \gamma_n J_1(\gamma_n)}{\left(\gamma_n^2 + h^2\right) J_0(\gamma_n)^2} = \frac{2hu_0}{\left(\gamma_n^2 + h^2\right) J_0(\gamma_n)}, \end{aligned}$$

so

$$u(r,z) = 2hu_0 \sum_{n=1}^{\infty} \frac{\exp(-\gamma_n z) J_0(\gamma_n r)}{\left(\gamma_n^2 + h^2\right) J_0(\gamma_n)}.$$

11. When we substitute $u(r, t) = R(r)\sin \omega t$ in the given partial differential equation and cancel the factor $\sin \omega t$, we get the ordinary differential equation

$$R'' + \frac{1}{r}R' + \left(\frac{\omega}{a}\right)^2 R = -\frac{F_0}{a^2}$$

The associated homogeneous equation is the Bessel equation of order zero with parameter ω/a. Hence it follows readily that the solution that is continuous at $r = 0$ is

$$R(r) = A J_0\left(\frac{\omega r}{a}\right) - \frac{F_0}{\omega^2}.$$

The condition $R(b) = 0$ yields $A = F_0 / \omega^2 J_0(\omega b / a)$, so it follows that the desired steady periodic solution is

$$u(r,t) = \frac{F_0}{\omega^2 J_0(\omega b / a)}\left[J_0\left(\frac{\omega r}{a}\right) - J_0\left(\frac{\omega b}{a}\right)\right]\sin \omega t.$$

12. When we substitute $y(x, t) = X(x)\sin \omega t$ in the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{g}{w}\frac{\partial}{\partial x}\left(wx\frac{\partial y}{\partial x}\right) = g\left(\frac{\partial y}{\partial x} + x\frac{\partial^2 y}{\partial x^2}\right)$$

and then cancel the $\sin \omega t$ factor, we get the ordinary differential equation

$$x^2X'' + xX' + \frac{\omega^2 x}{g}X = 0.$$

This is of the form of Equation (3) in Section 8.6 with $A = 1$, $B = 0$, $C = \omega^2/g$, and $q = 1$, so its general solution is given by

$$X(x) = A\,J_0\left(2\omega\sqrt{\frac{x}{g}}\right) + B\,Y_0\left(2\omega\sqrt{\frac{x}{g}}\right).$$

We choose $B = 0$ for continuity at $x = 0$, and the condition $X(L) = 0$ then requires that $2\omega\sqrt{L/g} = \gamma_n$, one of the roots $\{\gamma_n\}$ of the equation $J_0(x) = 0$. Hence the nth natural frequency of vibration of the hanging cable is

$$\omega_n = \frac{\gamma_n}{2}\sqrt{\frac{x}{g}}.$$

13. With $w(x) = wx$ and $h(x) = h$ (where w and h on the right are constants) the given partial differential equation

$$\frac{w(x)}{g}\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(w(x)h(x)\frac{\partial y}{\partial x}\right) \qquad (*)$$

reduces to

$$x\frac{\partial^2 y}{\partial t^2} = gh\left(\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\right).$$

When we substitute $y(x, t) = X(x)\cos \omega t$ we get the parametric Bessel equation

$$x^2X'' + xX' + \frac{\omega^2 x^2}{gh}X = 0$$

with bounded solution

$$X(x) = A\,J_0\left(\frac{\omega x}{\sqrt{gh}}\right).$$

The condition $X(L) = y_0$ implies that $A = y_0 / J_0\left(\omega L/\sqrt{gh}\right)$, so

$$y(x,t) = y_0\frac{J_0\left(\omega x/\sqrt{gh}\right)}{J_0\left(\omega L/\sqrt{gh}\right)}\cos \omega t.$$

14. With $w(x) = w$ and $h(x) = hx$ (with w and h being constants on the right) the partial differential equation in (*) above reduces to

$$\frac{\partial^2 y}{\partial t^2} = gh\left(\frac{\partial y}{\partial x} + x\frac{\partial^2 y}{\partial x^2}\right).$$

When we substitute $y(x, t) = X(x)\cos \omega t$ we get the ordinary differential equation

$$x^2 X'' + xX' + \frac{\omega^2 x}{gh} X = 0.$$

This has the form of Equation (3) in Section 8.6 with $A = 1$, $B = 0$, $C = \omega^2/gh$, and $q = 1$, so its (bounded) solution is given by

$$X(x) = AJ_0\left(2\omega\sqrt{\frac{x}{gh}}\right).$$

The condition $X(L) = y_0$ now implies that $A = y_0 / J_0\left(2\omega\sqrt{L/gh}\right)$, so

$$y(x,t) = y_0 \frac{J_0\left(2\omega\sqrt{x/gh}\right)}{J_0\left(2\omega\sqrt{L/gh}\right)}\cos\omega t.$$

15. With $w(x) = wx$ and $h(x) = hx$ (with w and h being constants on the right) the partial differential equation in (*) above reduces to

$$\frac{\partial^2 y}{\partial t^2} = gh\left(2\frac{\partial y}{\partial x} + x\frac{\partial^2 y}{\partial x^2}\right).$$

When we substitute $y(x, t) = X(x)\cos \omega t$ we get the ordinary differential equation

$$x^2 X'' + 2xX' + \frac{\omega^2 x}{gh} X = 0.$$

This has the form of Equation (3) in Section 8.6 with $A = 2$, $B = 0$, $C = \omega^2/gh$, and $q = 1$, so its (bounded) solution is given by

$$X(x) = \frac{A}{\sqrt{x}} J_1\left(2\omega\sqrt{\frac{x}{gh}}\right).$$

The condition $X(L) = y_0$ now implies that $A = y_0\sqrt{L} / J_1\left(2\omega\sqrt{L/gh}\right)$, so

$$y(x,t) = y_0\sqrt{\frac{L}{x}}\frac{J_1\left(2\omega\sqrt{x/gh}\right)}{J_1\left(2\omega\sqrt{L/gh}\right)}\cos\omega t.$$

16. With $\lambda = \alpha^2$ the general solution of the parametric Bessel equation of order 0 is

$$y(x) = AJ_0(\alpha x) + BY_0(\alpha x).$$

The endpoint conditions $y(a) = y(b) = 0$ yield the linear equations

$$A\ J_0(\alpha a) + B\ Y_0(\alpha a) = 0,$$
$$A\ J_0(\alpha b) + B\ Y_0(\alpha b) = 0$$

in A and B. In order for there to exist a non-trivial solution for A and B the coefficient determinant must vanish. Hence α must be one of the solutions $\{\gamma_n\}$ of the equation

$$J_0(ax)Y_0(bx) - J_0(bx)Y_0(ax) = 0. \qquad (\#)$$

With $\alpha = \gamma_n$, $A = Y_0(\gamma_n a)$ and $B = -J_0(\gamma_n a)$, both conditions above are satisfied and we have the eigenfunction

$$R_n(x) = Y_0(\gamma_n a)J_0(\gamma_n x) - J_0(\gamma_n a)Y_0(\gamma_n x).$$

17. Just as in Problem 1 above, substitution of $u(r,t) = R(r)T(t)$ in the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

yields the separation

$$\frac{T''}{a^2T} = \frac{R'' + \frac{1}{r}R'}{R} = \lambda = -\alpha^2.$$

The t-equation has general solution

$$T(t) = A\cos\alpha at + B\sin\alpha at,$$

and we choose $B = 0$, so that $T'(0) = 0$ (assuming, for instance, that the membrane is initially at rest). The r-equation can be written in the form

$$r^2R'' + rR' + \alpha^2r^2R = 0,$$

which is the parametric Bessel equation of order zero. By Problem 16, its solutions satisfying $R(a) = R(b) = 0$ are of the form $R_n(x) = Y_0(\gamma_n a)J_0(\gamma_n x) - J_0(\gamma_n a)Y_0(\gamma_n x)$ with $\alpha = \gamma_n$ being one of the positive roots of Equation (#) there. This leads to a formal series solution of the form

$$u(r,t) = \sum_{n=1}^{\infty} R_n(x)\left(A_n \cos\gamma_n at + B_n \sin\gamma_n at\right),$$

where the frequency of the nth term is $\omega_n = \gamma_n a = \gamma_n\sqrt{T/\rho}$.

18. We start with the substitution $u(r, t) = R(r)T(t)$ in the heat equation. The result is given in Equations (25) and (26):

$$r^2R'' + rR' + \alpha^2r^2R = 0, \qquad T' = -\alpha^2kT.$$

The first of these equations, together with the endpoint conditions

$$R(a) = R(b) = 0,$$

comprise the regular Sturm-Liouville problem of Problem 16. Hence its eigenvalues are given by $\alpha_n = \gamma_n$ where $\{\gamma_n\}$ are the positive roots of the equation in Eq.(41) in the text. The nth eigenfunction is the function $R_n(r)$ defined in (42). Finally the solution of $T_n' = -\gamma_n^2kT_n$ is

$$T_n(t) = \exp(-\gamma_n^2kt),$$

so we get a solution of the form

$$u(r,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\gamma_n^2kt\right)R_n(r).$$

19. We want to solve the boundary value problem

$$\frac{\partial^2u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2u}{\partial z^2} = 0 \qquad (a < r < 1,\ z > 0)$$

$$u(a,z) = u(b,z) = 0$$

$$u(r, z) \text{ bounded as } z \to \infty$$

$$u(r,0) = f(r).$$

Just as in Problem 5, the substitution $u(r, z) = R(r)Z(z)$ yields the equations

$$rR'' + R' + \alpha^2rR = 0, \qquad Z'' - \alpha^2Z = 0$$

with separation constant $\lambda = \alpha^2$. When we impose the conditions $R(a) = R(b) = 0$ on the r-equation here, we have the Sturm-Liouville problem of Problem 16, so $R(r)$ must be one of the eigenfunctions $\{R_n(r)\}$ corresponding to the positive roots $\{\gamma_n\}$ of Equation (#) there. The general solution of $Z'' = \gamma_n^2 Z$ is

$$Z_n(z) = A_n \exp(-\gamma_n z) + B_n \exp(\gamma_n z),$$

and we choose $B_n = 0$ so that $Z_n(z)$ will be bounded as $z \to \infty$. Thus we obtain a solution of the form

$$u(r,z) = \sum_{n=1}^{\infty} c_n \exp(-\gamma_n z) R_n(r)$$

with the coefficients $\{c_n\}$ calculated as in Problem 18.

SECTION 10.5

HIGHER-DIMENSIONAL PHENOMENA

This section provides the interested student with an opportunity to study several applications at greater depth than is afforded by the usual textbook exercises. The problem sets outlined in Section 10.5 can serve as the basis for several fairly substantial computational projects. Because these problem sets and projects are rather heavily annotated in the text, further outlines of solutions are not included in this manual. However, additional discussion — particularly regarding computer implementations — may be found in the applications manual that accompanies the text.