CHAPTER 4

INTRODUCTION TO SYSTEMS OF DIFFERENTIAL EQUATIONS

This chapter bridges the gap between the treatment of a single differential equation in Chapters 1-3 and the comprehensive treatment of linear and nonlinear systems in Chapters 5-6. It also is designed to offer some flexibility in the treatment of linear systems, depending on the background in linear algebra that students are assumed to have—Sections 4.1 and 4.2 can stand alone as a very brief introduction to linear systems without the use of linear algebra and matrices. The final Section 4.3 of this chapter extends to systems the numerical approximation techniques of Chapter 2.

SECTION 4.1

FIRST-ORDER SYSTEMS AND APPLICATIONS

1. Let $x_1 = x$ and $x_2 = x_1' = x'$, so that $x_2' = x'' = -7x - 3x' + t^2$. Equivalent system:

$$x_1' = x_2, \quad x_2' = -7x_1 - 3x_2 + t^2 .$$

2. Let $x_1 = x$ and $x_2 = x_1' = x'$, so that $x_2' = x'' = -4x + x^3$. Equivalent system:

$$x_1' = x_2, \quad x_2' = -4x_1 + x_1^3 .$$

3. Let $x_1 = x$ and $x_2 = x_1' = x'$, so that $x_2' = x'' = -26x - 2x' + 82\cos 4t$. Equivalent system:

$$x_1' = x_2, \quad x_2' = -26x_1 - 2x_2 + 82\cos 4t .$$

4. Let $x_1 = x$, $x_2 = x_1' = x'$, and $x_3 = x_2' = x''$, so that $x_3' = x''' = 2x'' - x' + 1 + te^t$. Equivalent system:

$$x_1' = x_2, \quad x_2' = x_3, \quad x_3' = 2x_3 - x_2 + 1 + te^t .$$

5. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, and $x_4 = x_3' = x'''$, so that $x_4' = x^{(4)} = -3x'' - x + e^t \sin 2t$. Equivalent system:

$$x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4, \quad x_4' = -3x_2 - x_1 + e^t \sin 2t .$$

6. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, and $x_4 = x_3' = x'''$, so that $x_4' = x^{(4)} = x + 3x' - 6x'' + \cos 3t$. Equivalent system:

$$x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4, \quad x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t .$$

7. Let $x_1 = x$ and $x_2 = x_1' = x'$, so that $x_2' = x'' = \dfrac{\left(1-t^2\right)x - tx'}{t^2}$. Equivalent system:

$$x_1' = x_2, \quad t^2 x_2' = \left(1-t^2\right)x_1 - tx_2 .$$

8. Let $x_1 = x$, $x_2 = x_1' = x'$, and $x_3 = x_2' = x''$, so that $x_3' = x''' = \dfrac{1}{t^3}\left(-5x - 3tx' - 2t^2x'' + \ln t\right)$. Equivalent system:

$$x_1' = x_2, \quad x_2' = x_3, \quad t^3 x_3' = -5x_1 - 3tx_2 + 2t^2x_3 + \ln t .$$

9. Let $x_1 = x$, $x_2 = x_1' = x'$, and $x_3 = x_2' = x''$, so that $x_3' = x''' = \left(x'\right)^2 + \cos x$. Equivalent system:

$$x_1' = x_2, \quad x_2' = x_3, \quad x_3' = {x_2}^2 + \cos x_1 .$$

10. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, and $y_2 = y_1' = y'$, so that $x_2' = x'' = 5x - 4y$ and $y_2' = y'' = -4x + 5y$. Equivalent system:

$$x_1' = x_2, \quad x_2' = 5x_1 - 4y_1, \quad y_1' = y_2, \quad y_2' = -4x_1 + 5y_1 .$$

11. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, and $y_2 = y_1' = y'$, so that $x_2' = x'' = -\dfrac{kx}{\left(x^2+y^2\right)^{3/2}}$ and

$y_2' = y'' = -\dfrac{ky}{\left(x^2+y^2\right)^{3/2}}$. Equivalent system:

$$x_1' = x_2, \quad x_2' = -\frac{kx_1}{\left(x_1^2+y_1^2\right)^{3/2}}, \quad y_1' = y_2, \quad y_2' = -\frac{ky_1}{\left(x_1^2+y_1^2\right)^{3/2}} .$$

12. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, and $y_2 = y_1' = y'$, so that $x_2' = x'' = \dfrac{2}{3}y'$ and

$y_2' = y'' = -\dfrac{2}{3}x'$. Equivalent system:

$$x_1' = x_2, \quad x_2' = \frac{2}{3}y_2, \quad y_1' = y_2, \quad y_2' = -\frac{2}{3}x_2 .$$

13. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, and $y_2 = y_1' = y'$, so that $x_2' = x'' = -75x + 25y$ and $y_2' = y'' = 50x - 50y + 50\cos 5t$. Equivalent system:

$$x_1' = x_2, \quad x_2' = -75x_1 + 25y_1, \quad y_1' = y_2, \quad y_2' = 50x_1 - 50y_1 + 50\cos 5t.$$

14. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, and $y_2 = y_1' = y'$, so that $x_2' = x'' = -4x + 2y - 3x'$ and $y_2' = y'' = 3x - y - 2y' + \cos t$. Equivalent system:

$$x_1' = x_2, \quad x_2' = -4x_1 + 2y_1 - 3x_2, \quad y_1' = y_2, \quad y_2' = 3x_1 - y_1 - 2y_2 + \cos t.$$

15. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, $y_2 = y_1' = y'$, $z_1 = z$, and $z_2 = z_1' = z'$, so that $x_2' = x'' = 3x - y + 2z$, $y_2' = y'' = x + y - 4z$, and $z_2' = z'' = 5x - y - z$. Equivalent system:

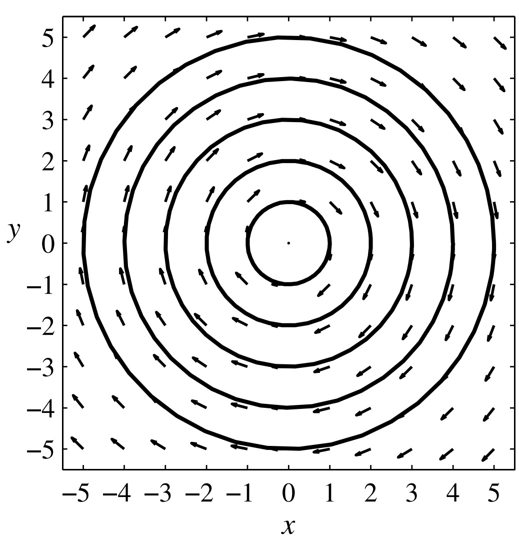
$$\begin{aligned} x_1' &= x_2, & x_2' &= 3x_1 - y_1 + 2z_1 \\ y_1' &= y_2, & y_2' &= x_1 + y_1 - 4z_1 \\ z_1' &= z_2, & z_2' &= 5x_1 - y_1 - z_1 \end{aligned}$$

16. Let $x_1 = x$, $x_2 = x_1' = x'$, $y_1 = y$, and $y_2 = y_1' = y'$, so that $x_2' = x'' = x(1-y)$ and $y_2' = y'' = y(1-x)$. Equivalent system:

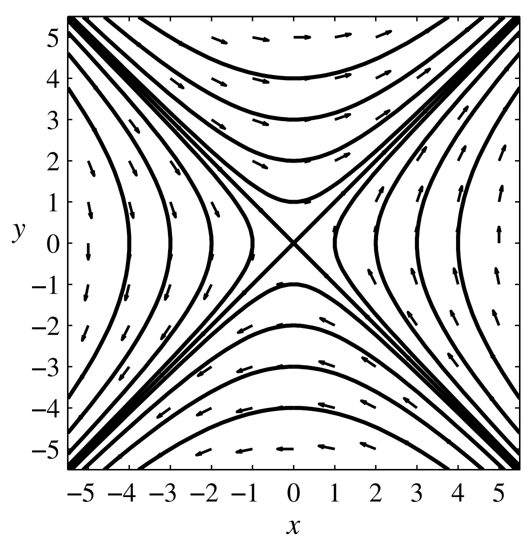
$$x_1' = x_2, \quad x_2' = x_1(1-y_1), \quad y_1' = y_2, \quad y_2' = y_1(1-x_1).$$

17. The computation $x'' = y' = -x$ yields the single linear second-order equation $x'' + x = 0$ with characteristic equation $r^2 + 1 = 0$ and general solution $x(t) = A\cos t + B\sin t$. Then the original first equation $y = x'$ gives $y(t) = B\cos t - A\sin t$. The figure shows a direction field and typical solution curves (obviously circles?) for the given system.

Problem 17



Problem 18



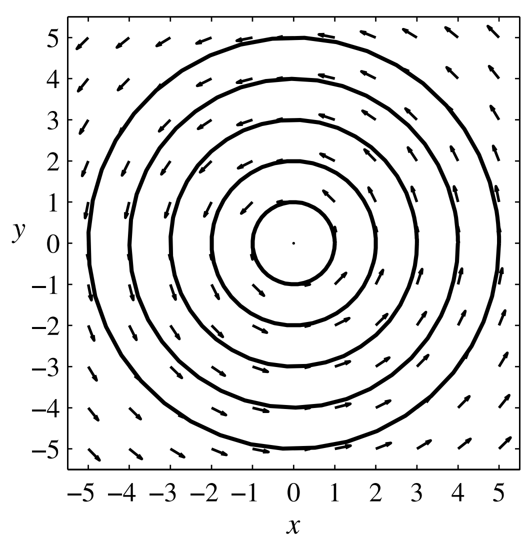
18. The computation $x'' = y' = x$ yields the single linear second-order equation $x'' - x = 0$ with characteristic equation $r^2 - 1 = 0$ and general solution $x(t) = Ae^t + Be^{-t}$. Then the original first equation $y = x'$ gives $y(t) = Ae^t - Be^{-t}$. The figure shows a direction field and some typical solution curves of this system. It appears that the typical solution curve is a branch of a hyperbola.

19. The computation $x'' = -2y' = -4x$ yields the single linear second-order equation $x'' + 4x = 0$ with characteristic equation $r^2 + 4 = 0$ and general solution $x(t) = A\cos 2t + B\sin 2t$. Then the original first equation $y = -\frac{1}{2}x'$ gives $y(t) = -B\cos 2t + A\sin 2t$. Finally, the condition $x(0) = 1$ implies that $A = 1$, and then the condition $y(0) = 0$ gives $B = 0$. Hence the desired particular solution is given by

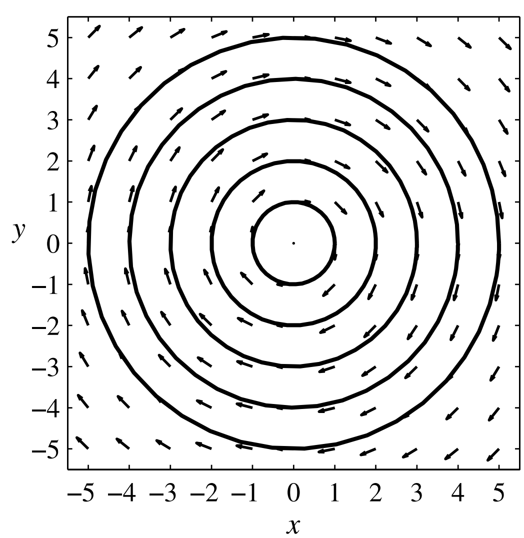
$$x(t) = \cos 2t, \quad y(t) = \sin 2t.$$

The figure shows a direction field and some typical circular solution curves for the given system.

Problem 19



Problem 20



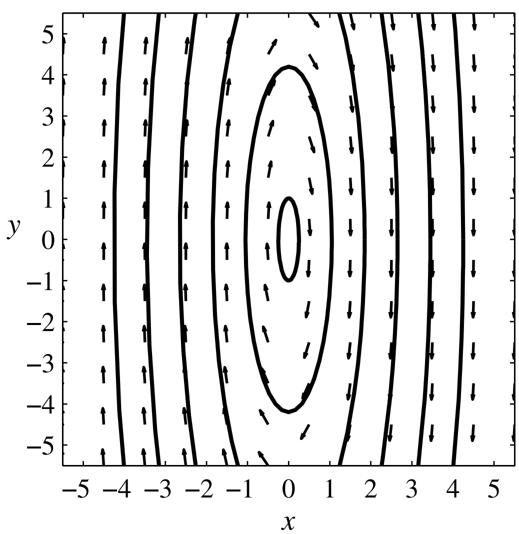
20. The computation $x'' = 10y' = -100x$ yields the single linear second-order equation $x'' + 100x = 0$ with characteristic equation $r^2 + 100 = 0$ and general solution $x(t) = A\cos 10t + B\sin 10t$. Then the original first equation $y = \frac{1}{10}x'$ gives $y(t) = B\cos 10t - A\sin 10t$. Finally, the condition $x(0) = 3$ implies that $A = 3$, and then the condition $y(0) = 4$ gives $B = 4$. Hence the desired particular solution is given by

$$x(t) = 3\cos 10t + 4\sin 10t, \quad y(t) = 4\cos 10t - 3\sin 10t.$$

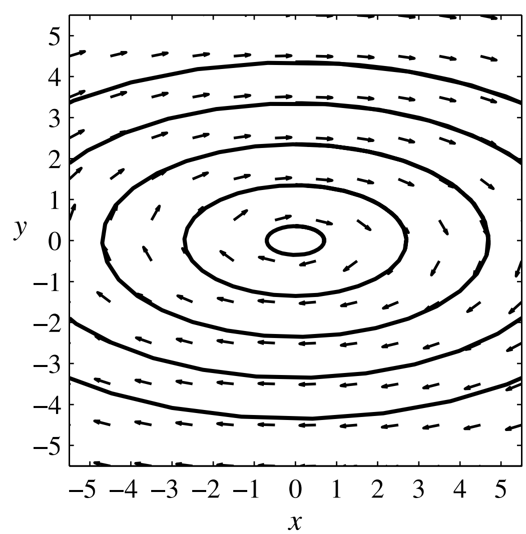
The typical solution curve is a circle, as the figure suggests.

21. The computation $x'' = \frac{1}{2}y' = -4x$ yields the single linear second-order equation $x'' + 4x = 0$ with characteristic equation $r^2 + 4 = 0$ and general solution $x(t) = A\cos 2t + B\sin 2t$. Then the original first equation $y = 2x'$ gives $y(t) = 4B\cos 2t - 4A\sin 2t$. The figure shows a direction field and some typical elliptical solution curves.

Problem 21



Problem 22



22. The computation $x'' = 8y' = -16x$ yields the single linear second-order equation $x'' + 16x = 0$ with characteristic equation $r^2 + 16 = 0$ and general solution $x(t) = A\cos 4t + B\sin 4t$. Then the original first equation $y = \frac{1}{8}x'$ gives $y(t) = \frac{B}{2}\cos 4t - \frac{A}{2}\sin 4t$. The typical solution curve is an ellipse. The figure shows a direction field and some typical solution curves.

23. The computation $x'' = y' = 6x - y = 6x - x'$ yields the single linear second-order equation $x'' + x' - 6x = 0$ with characteristic equation $r^2 + r - 6 = 0$, characteristic roots $r = -3$ and 2, and general solution $x(t) = Ae^{-3t} + Be^{2t}$. Then the original first equation $y = x'$ gives $y(t) = -3Ae^{-3t} + 2Be^{2t}$. Finally, the initial conditions

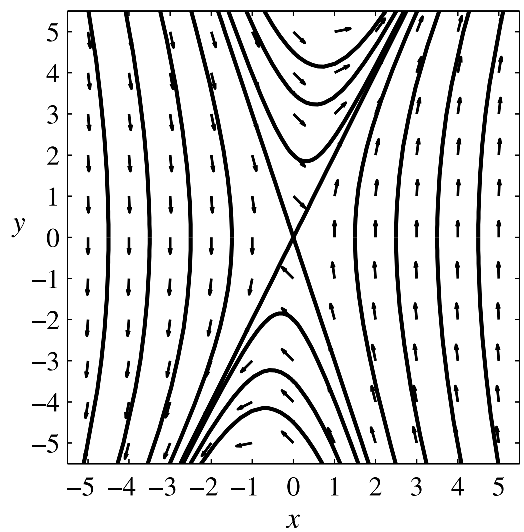
$$x(0) = A + B = 1, \quad y(0) = -3A + 2B = 2$$

imply that $A = 0$ and $B = 1$, so the desired particular solution is given by

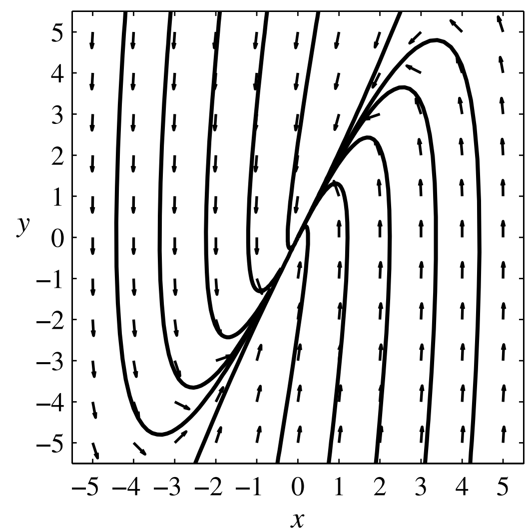
$$x(t) = e^{2t}, \quad y(t) = 2e^{2t}.$$

The figure shows a direction field and some typical solution curves.

Problem 23



Problem 24



24. The computation $x'' = -y' = -10x + 7y = -10x - 7x'$ yields the single linear second-order equation $x'' + 7x' + 10x = 0$ with characteristic equation $r^2 + 7r + 10 = 0$, characteristic roots $r = -2$ and -5, and general solution $x(t) = Ae^{-2t} + Be^{-5t}$. Then the original first equation $y = -x'$ gives $y(t) = 2Ae^{-2t} + 5Be^{-5t}$. Finally, the initial conditions

$$x(0) = A + B = 2, \quad y(0) = 2A + 5B = -7$$

imply that $A = \frac{17}{3}$ and $B = -\frac{11}{3}$, so the desired particular solution is given by

$$x(t) = \frac{1}{3}\left(17e^{-2t} - 11e^{-5t}\right), \quad y(t) = \frac{1}{3}\left(34e^{-2t} - 55e^{-5t}\right).$$

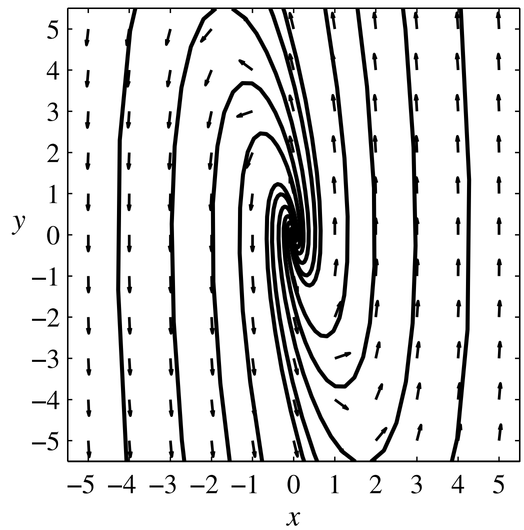
It appears that the typical solution curve is tangent to the straight line $y = 2x$. The figure shows a direction field and some typical solution curves.

25. The computation $x'' = -y' = -13x - 4y = -13x + 4x'$ yields the single linear second-order equation $x'' - 4x' + 13x = 0$ with characteristic equation $r^2 - 4r + 13 = 0$ and characteristic roots $r = 2 \pm 3i$; hence the general solution is $x(t) = e^{2t}\left(A\cos 3t + B\sin 3t\right)$. The initial condition $x(0) = 0$ then gives $A = 0$, so $x(t) = Be^{2t}\sin 3t$. Then the original first equation $y = -x'$ gives $y(t) = -e^{2t}\left(3B\cos 3t + 2B\sin 3t\right)$. Finally, the initial condition $y(0) = 3$ gives $B = -1$, so the desired particular solution is given by

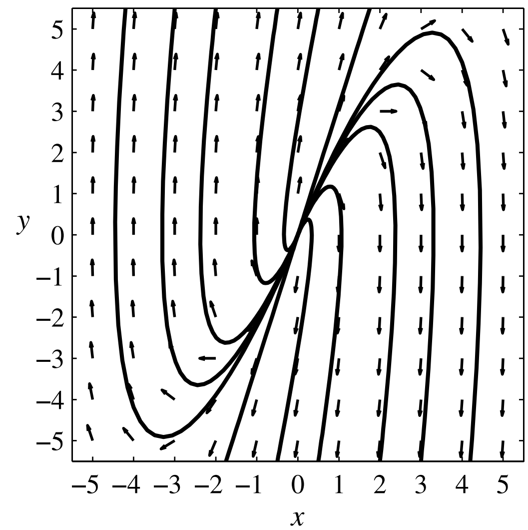
$$x(t) = -e^{2t}\sin 3t, \quad y(t) = e^{2t}\left(3\cos 3t + 2\sin 3t\right).$$

The figure shows a direction field and some typical solution curves.

Problem 25



Problem 26



26. The computation $x'' = y' = -9x + 6y = -9x + 6x'$ yields the single linear second-order equation $x'' - 6x' + 9x = 0$ with characteristic equation $r^2 - 6r + 9 = 0$ and repeated characteristic root $r = 3, 3$, so its general solution is given by $x(t) = (A + Bt)e^{3t}$. Then the original first equation $y = x'$ gives $y(t) = (3A + B + 3Bt)e^{3t}$. It appears that the typical solution curve is tangent to the straight line $y = 3x$. The figure shows a direction field and some typical solution curves.

27. **(a)** Substituting the general solution found in Problem 17 we get

$$\begin{aligned} x^2 + y^2 &= (A\cos t + B\sin t)^2 + (B\cos t - A\sin t)^2 \\ &= (A^2 + B^2)(\cos^2 t + \sin^2 t) \\ &= (A^2 + B^2), \end{aligned}$$

or $x^2 + y^2 = C^2$, the equation of a circle of radius $C = \sqrt{A^2 + B^2}$.

(b) Substituting the general solution found in Problem 18, we get

$$x^2 - y^2 = (Ae^t + Be^{-t})^2 - (Ae^t - Be^{-t})^2 = 4AB,$$

the equation of a hyperbola.

28. **(a)** Substituting the general solution found in Problem 19 we get

$$\begin{aligned} x^2 + y^2 &= (A\cos 2t + B\sin 2t)^2 + (-B\cos 2t + A\sin 2t)^2 \\ &= (A^2 + B^2)(\cos^2 2t + \sin^2 2t) \\ &= (A^2 + B^2), \end{aligned}$$

or $x^2 + y^2 = C^2$, the equation of a circle of radius $C = \sqrt{A^2 + B^2}$.

(b) Substituting the general solution found in Problem 21 we get

$$\begin{aligned} 16x^2 + y^2 &= 16\left(A\cos 2t + B\sin 2t\right)^2 + \left(4B\cos 2t - 4A\sin 2t\right)^2 \\ &= 16\left(A^2 + B^2\right)\left(\cos^2 2t + \sin^2 2t\right) \\ &= 16\left(A^2 + B^2\right), \end{aligned}$$

or $16x^2 + y^2 = C^2$, the equation of an ellipse with semi-axes 1 and 4.

29. When we solve Equations (20) and (21) in the text for e^{-t} and e^{2t} we get $2x - y = 3Ae^{-t}$ and $x + y = 3Be^{2t}$. Hence

$$(2x - y)^2 (x + y) = \left(3Ae^{-t}\right)^2 \cdot 3Be^{2t} = 27A^2B = C.$$

Clearly $y = 2x$ or $y = -x$ if $C = 0$, and expansion gives the equation $4x^3 - 3xy^2 + y^3 = C$.

30. Looking at Fig. 4.1.11 in the text, we see that the first spring is stretched by x_1, the second spring is stretched by $x_2 - x_1$, and the third spring is compressed by x_2. Hence Newton's second law gives $m_1 x_1'' = -k_1(x_1) + k_2(x_2 - x_1)$ and $m_2 x_2'' = -k_2(x_2 - x_1) - k_3(x_2)$.

31. Looking at Fig. 4.1.12 in the text, we see that

$$my_1'' = -T\sin\theta_1 + T\sin\theta_2 \approx -T\tan\theta_1 + T\tan\theta_2 = -\frac{Ty_1}{L} + \frac{T(y_2 - y_1)}{L}$$

and

$$my_2'' = -T\sin\theta_2 - T\sin\theta_3 \approx -T\tan\theta_2 - T\tan\theta_3 = -\frac{T(y_2 - y_1)}{L} - \frac{Ty_2}{L}.$$

We get the desired equations when we multiply each of these equations by $\frac{L}{T}$ and set $k = \frac{mL}{T}$.

32. The concentration of salt in tank *i* is $c_i = \frac{x_i}{100}$ for $i = 1, 2, 3, \ldots$ and each inflow-outflow rate is $r = 10$. Hence

$$x_1' = -rc_1 + rc_3 = \frac{1}{10}(-x_1 + x_3),$$
$$x_2' = +rc_1 - rc_2 = \frac{1}{10}(x_1 - x_2),$$
$$x_3' = +rc_2 - rc_3 = \frac{1}{10}(x_2 - x_3).$$

33. We apply Kirchhoff's law to each loop in Figure 4.1.14 in the text, and immediately get the equations

$$2(I_1' - I_2') + 50I_1 = 100\sin 60t, \quad 2(I_2' - I_1') + 25I_2 = 0\,.$$

34. First we apply Kirchhoff's law to each loop in Figure 4.1.14 in the text, denoting by Q the charge on the capacitor, and get the equations

$$50I_1 + 1000Q = 100, \quad 25I_2 - 1000Q = 0\,.$$

Then we differentiate each equation and substitute $Q' = I_1 - I_2$ to get the system

$$I_1' = -20(I_1 - I_2), \quad I_2' = 40(I_1 - I_2)\,.$$

35. If θ is the polar angular coordinate of the point (x, y) and we write $F = \dfrac{k}{x^2 + y^2} = \dfrac{k}{r^2}$, then Newton's second law gives

$$mx'' = -F\cos\theta = -\frac{k}{r^2}\cdot\frac{x}{r} = -\frac{kx}{r^3}, \quad my'' = -F\sin\theta = -\frac{k}{r^2}\cdot\frac{y}{r} = -\frac{ky}{r^3}\,.$$

36. If we write (x', y') for the velocity vector and $v = \sqrt{(x')^2 + (y')^2}$ for the speed, then $\left(\dfrac{x'}{v}, \dfrac{y'}{v}\right)$ is a unit vector pointing in the direction of the velocity vector, and so the components of the air resistance force F_r are given by

$$F_r = -kv^2\left(\frac{x'}{v}, \frac{y'}{v}\right) = (-kvx', -kvy')\,.$$

37. If $\mathbf{r} = (x, y, z)$ is the particle's position vector, then Newton's law $m\mathbf{r}'' = \mathbf{F}$ gives

$$m\mathbf{r}'' = q\mathbf{v}\times\mathbf{B} = q\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & z' \\ 0 & 0 & B \end{vmatrix} = +qBy'\mathbf{i} - qBx'\mathbf{j} = qB(-y', x', 0)\,.$$

SECTION 4.2

THE METHOD OF ELIMINATION

1. The second differential equation $y' = 2y$ has the exponential solution

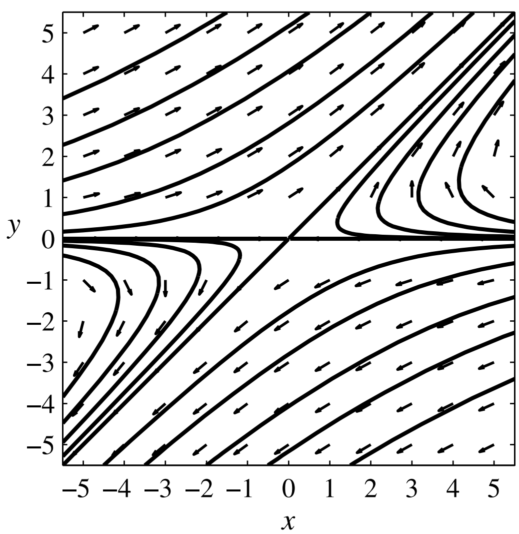
$$y(t) = c_2 e^{2t}.$$

Substitution in the first differential equation gives the linear first-order equation $x' + x = 3c_2 e^{2t}$ with integrating factor $\rho = e^t$. Solution of this equation in the usual way gives

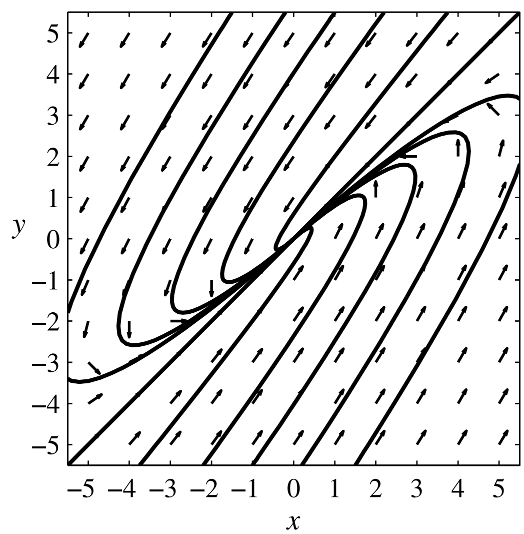
$$x(t) = e^{-t}\left(c_1 + c_2 e^{3t}\right) = c_1 e^{-t} + c_2 e^{2t}.$$

The figure shows a direction field and some typical solution curves.

Problem 1



Problem 2



2. From the first differential equation we get $y = \frac{1}{2}(x - x')$, so $y = \frac{1}{2}(x' - x'')$. Substitution of these expressions for y and y' into the second differential equation yields the second-order equation $x'' + 2x' + x = 0$ with general solution

$$x(t) = (c_1 + c_2 t)e^{-t}.$$

Substitution in $y = \frac{1}{2}(x - x')$ now yields

$$y(t) = \left(c_1 - \frac{1}{2}c_2 + c_2 t\right)e^{-t}.$$

The figure shows a direction field and some typical solution curves.

3. From the first differential equation we get $y = \frac{1}{2}(3x + x')$, so $y' = \frac{1}{2}(3x' + x'')$. Substitution of these expressions for y and y' into the second differential equation yields the second-order equation $x'' - x' - 6x = 0$ with general solution

$$x(t) = c_1 e^{-2t} + c_2 e^{3t}.$$

Substitution in $y = \frac{1}{2}(3x + x')$ now yields

$$y(t) = \frac{1}{2} c_1 e^{-2t} + 3c_2 e^{3t}.$$

Imposition of the initial conditions $x(0) = 0$, $y(0) = 2$ now gives the equations

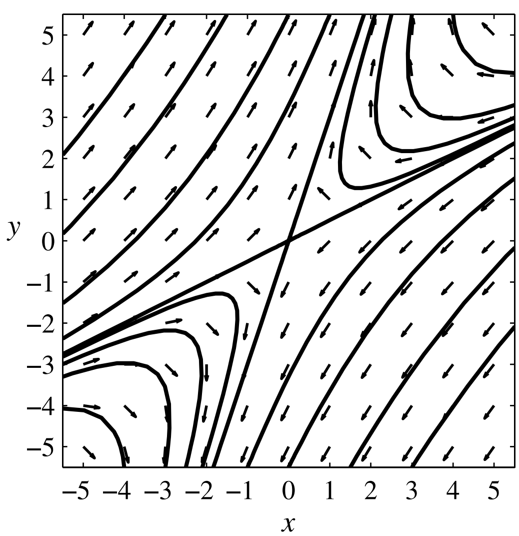
$$c_1 + c_2 = 0, \quad \frac{1}{2} c_1 + 3c_2 = 2$$

with solution $c_1 = -\frac{4}{5}$, $c_2 = \frac{4}{5}$. These coefficients give the desired particular solution

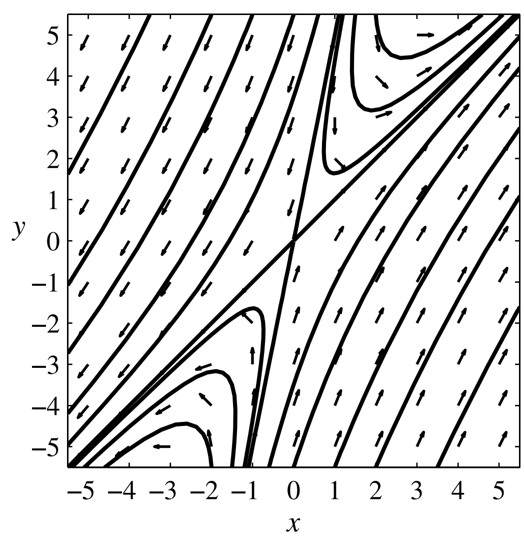
$$x(t) = \frac{4}{5}\left(e^{3t} - e^{-2t}\right), \quad y(t) = \frac{2}{5}\left(6e^{3t} - e^{-2t}\right).$$

The figure shows a direction field and some typical solution curves.

Problem 3



Problem 4



4. Substitution of $y = 3x - x'$ and $y' = 3x' - x''$ —from the first equation—into the second equation yields the second-order equation $x'' - 4x = 0$ with general solution

$$x(t) = c_1 e^{2t} + c_2 e^{-2t}.$$

Substitution of this solution in $y = 3x - x'$ gives

$$y(t) = c_1 e^{2t} + 5c_2 e^{-2t}.$$

The initial conditions yield the equations

$$c_1 + c_2 = 1, \quad c_1 + 5c_2 = -1$$

with solution $c_1 = \frac{3}{2}$, $c_2 = -\frac{1}{2}$. Hence the desired particular solution is

$$x(t) = \frac{1}{2}\left(3e^{2t} - e^{-2t}\right), \quad y(t) = \frac{1}{2}\left(3e^{2t} - 5e^{-2t}\right).$$

The figure shows a direction field and some typical solution curves.

5. Substitution of $y = -\frac{1}{4}(x' + 3x)$ and $y' = -\frac{1}{4}(x'' + 3x')$ —from the first equation—into the second equation yields the second-order equation $x'' + 2x' + 5x = 0$ with general solution

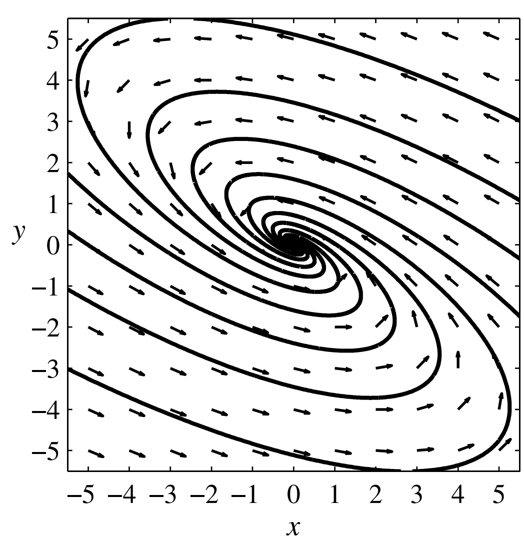
$$x(t) = e^{-t}\left(c_1 \cos 2t + c_2 \sin 2t\right).$$

Substitution of this solution in $y = -\frac{1}{4}(x' + 3x)$ gives

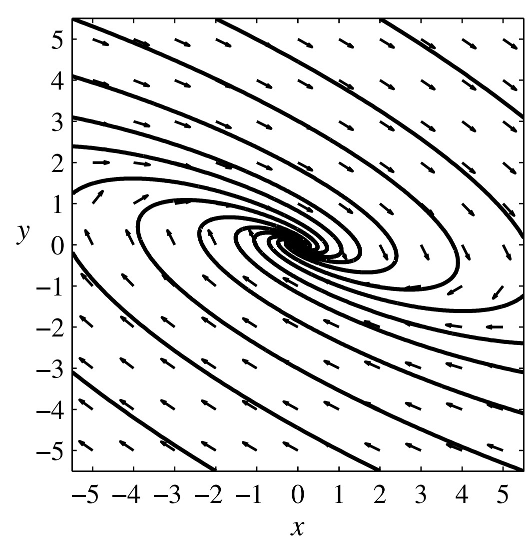
$$y(t) = \frac{1}{2}e^{-t}\left[-(c_1 + c_2)\cos 2t + (c_1 - c_2)\sin 2t\right].$$

The figure shows a direction field and some typical solution curves.

Problem 5



Problem 6



6. Substitution of $y = \frac{1}{9}(x' - x)$ and $y' = \frac{1}{9}(x'' - x')$ —from the first equation—into the second equation yields the second-order equation $x'' + 4x' + 13x = 0$ with general solution

$$x(t) = e^{-2t}\left(c_1 \cos 3t + c_2 \sin 3t\right).$$

Substitution in $y = \frac{1}{9}(x' - x)$ now yields

$$y(t) = \frac{1}{3}e^{-2t}\left[(-c_1 + c_2)\cos 3t + (-c_1 - c_2)\sin 3t\right].$$

Imposition of the initial conditions $x(0) = 3$, $y(0) = 2$ now gives the equations

$$c_1 = 3, \quad -\frac{c_1}{3} + \frac{c_2}{3} = 2$$

with solution $c_1 = 3$, $c_2 = 9$. These coefficients give the desired particular solution

$$x(t) = e^{-2t}\left(3\cos 3t + 9\sin 3t\right), \quad y(t) = e^{-2t}\left(2\cos 3t - 4\sin 3t\right).$$

The figure shows a direction field and some typical solution curves.

7. Substitution of $y = x' - 4x - 2t$ and $y' = x'' - 4x' - 2$ —from the first equation—into the second equation yields the nonhomogeneous second-order equation $x'' - 5x' + 6x = 2 - 2t$. Substitution of the trial solution $x_p = A + Bt$ yields $A = \frac{1}{18}$, $B = -\frac{1}{3}$, so $x_p = \frac{1}{18} - \frac{t}{3}$. Hence the general solution for x is

$$x(t) = c_1 e^{2t} + c_2 e^{3t} - \frac{t}{3} + \frac{1}{18}.$$

Substitution in $y = x' - 4x - 2t$ now yields

$$y(t) = -2c_1 e^{2t} - c_2 e^{3t} - \frac{2}{3}t - \frac{5}{9}.$$

8. Substitution of $y = x' - 2x$ and $y' = x'' - 2x'$ —from the first equation—into the second equation yields the nonhomogeneous second-order equation $x'' - 4x' + 3x = -e^{2t}$. Substitution of the trial solution yields $A = 1$, so $x_p = e^{2t}$. Hence the general solution for x is

$$x(t) = c_1 e^{t} + c_2 e^{3t} + e^{2t}.$$

Substitution in $y = x' - 2x$ now yields

$$y(t) = -c_1 e^{t} + c_2 e^{3t}.$$

9. Substitution of $y = \frac{1}{3}(-x' + 2x + 2\sin 2t)$ and $y' = \frac{1}{3}(-x'' + 2x' + 4\cos 2t)$ —from the first equation—into the second equation yields the second-order equation $x'' - x = 7\cos 2t + 4\sin 2t$ with general solution

$$x(t) = c_1 e^{-t} + c_2 e^{t} - \frac{1}{5}(7\cos 2t + 4\sin 2t).$$

Substitution in $y = \frac{1}{3}(-x' + 2x + 2\sin 2t)$ now yields

$$y(t) = c_1 e^{-t} + \frac{1}{3}c_2 e^{t} - \frac{1}{5}(2\cos 2t + 4\sin 2t).$$

10. First we solve the given equations for the normal-form first-order equations

$$x' = 2x + y, \quad y' = x + 2y.$$

Substitution of $y = x' - 2x$ and $y' = x'' - 2x'$ —from the first equation—into the second equation yields the second-order equation $x'' - 4x' + 3x = 0$ with general solution

$$x(t) = c_1 e^{t} + c_2 e^{3t}.$$

Substitution in $y = x' - 2x$ now yields

$$y(t) = -c_1 e^{t} + c_2 e^{3t}.$$

Imposition of the initial conditions $x(0) = 1$, $y(0) = -1$ now gives the equations

$$c_1 + c_2 = 1, \quad -c_1 + c_2 = -1$$

with solution $c_1 = 1$, $c_2 = 0$. These coefficients give the desired particular solution

$$x(t) = e^{t}, \quad y(t) = -e^{t}.$$

11. First we solve the given equations for the normal-form first-order equations

$$x' = 3x - 9y + e^{-t} + 2e^{t}, \quad y' = 2x - 3y + \frac{1}{2}e^{-t} + \frac{3}{2}e^{t}.$$

Substitution of $y = \frac{1}{9}(-x' + 3x + e^{-t} + 2e^{t})$ and $y' = \frac{1}{9}(-x'' + 3x' - e^{-t} + 2e^{t})$ —from the first equation—into the second equation yields the nonhomogeneous second-order equation $x'' + 9x = -\frac{1}{2}(5e^{-t} + 11e^{t})$. Substitution of the trial solution $x_p = Ae^{-t} + Be^{t}$ yields $A = -\frac{1}{4}$, $B = -\frac{11}{20}$, so $x_p = -\frac{1}{4}e^{-t} - \frac{11}{20}e^{t}$. Hence the general solution for *x* is

$$x(t) = c_1\cos 3t + c_2\sin 3t - \frac{1}{4}e^{-t} - \frac{11}{20}e^{t}.$$

Substitution in $y = \frac{1}{9}(-x' + 3x + e^{-t} + 2e^{t})$ now yields

$$y(t) = \frac{1}{3}(c_1 - c_2)\cos 3t + \frac{1}{3}(c_1 + c_2)\sin 3t + \frac{1}{10}e^{t}.$$

12. The first equation yields $y = \frac{1}{2}(x'' - 6x)$, so $y'' = \frac{1}{2}\left(x^{(4)} - 6x''\right)$. Substitution in the second equation yields

$$x^{(4)} - 13x'' + 36x = 0.$$

The characteristic equation is $r^4 - 13r^2 + 36 = \left(r^2 - 4\right)\left(r^2 - 9\right) = 0$, so the general solution for x is

$$x(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 e^{3t} + c_4 e^{-3t}.$$

Substitution in $y = \frac{1}{2}(x'' - 6x)$ now gives

$$y(t) = -c_1 e^{2t} - c_2 e^{-2t} + \frac{3}{2} c_3 e^{3t} + \frac{3}{2} c_4 e^{-3t}.$$

13. The first equation yields $y = \frac{1}{2}(x'' + 5x)$, so $y'' = \frac{1}{2}\left(x^{(4)} + 5x''\right)$. Substitution in the second equation yields

$$x^{(4)} + 13x'' + 36x = 0.$$

The characteristic equation is $r^4 + 13r^2 + 36 = \left(r^2 + 4\right)\left(r^2 + 9\right) = 0$, so the general solution for x is

$$x(t) = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 3t + b_2 \sin 3t.$$

Substitution in $y = \frac{1}{2}(x'' + 5x)$ now gives

$$y(t) = \frac{1}{2} a_1 \cos 2t + \frac{1}{2} a_2 \cos 2t - 2b_1 \cos 3t - 2b_2 \sin 3t.$$

14. The first equation $x'' + 4x = \sin t$ has complementary function $x_c = c_1 \cos 2t + c_2 \sin 2t$, and substitution of the trial solution $x = A \sin t$ yields the particular solution $x_p = \frac{1}{3} \sin t$. Hence the general solution for x is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \sin t.$$

Substitution in the second differential equation gives the equation

$$y'' + 8y = 4c_1 \cos 2t + 4c_2 \sin 2t + \frac{4}{3} \sin t$$

with complementary function $y_c = c_3\cos\left(2\sqrt{2}t\right) + c_4\sin\left(2\sqrt{2}t\right)$. Substitution of the trial solution $y_p = A\cos 2t + B\sin 2t + C\sin t$ now yields $A = c_1$, $B = c_2$, and $C = \dfrac{4}{21}$, so the general solution for y is

$$y(t) = c_1\cos 2t + c_2\sin 2t + c_3\cos\left(2\sqrt{2}t\right) + c_4\sin\left(2\sqrt{2}t\right) + \frac{4}{21}\sin t.$$

15. If we write the given differential equations in operator notation as

$$\begin{aligned}(D^2-2)x - 3Dy &= 0\\ 3Dx + (D^2-2)y &= 0,\end{aligned}$$

then we see that the system has operational determinant

$$(D^2-2)^2 + 9D^2 = D^4 + 5D^2 + 4 = (D^2+1)(D^2+4).$$

Therefore (as in Example 3) we see that x satisfies the fourth-order differential equation $(D^2+1)(D^2+4)x = 0$ with characteristic equation $(r^2+1)(r^2+4) = 0$ and general solution

$$x(t) = a_1\cos t + a_2\sin t + b_1\cos 2t + b_2\sin 2t.$$

Similarly, the general solution for y is of the form

$$y(t) = c_1\cos t + c_2\sin t + d_1\cos 2t + d_2\sin 2t.$$

Now, substitution of these two general solutions in the first equation $x'' - 3y' - 2x = 0$ and collection of coefficients gives

$$(-3a_1 - 3c_2)\cos t + (3c_1 - 3a_2)\sin t + (-6b_1 - 6d_2)\cos 2t + (3d_1 - 3b_2)\sin 2t = 0.$$

Thus we see finally that $c_1 = a_2$, $c_2 = -a_1$, $d_1 = b_2$, $d_2 = -b_1$. Hence

$$y(t) = a_2\cos t - a_1\sin t + b_2\cos 2t - b_1\sin 2t.$$

16. In operational form our system is

$$\begin{aligned}(D^2-4)x \quad + \quad 13Dy \quad &= \quad 6\sin t,\\ -2Dx \quad + \quad (D^2-9)y \quad &= \quad 0.\end{aligned}$$

When we operate on the first equation with $D^2 - 9$, on the second with $13D$, and subtract, the result is

$$(D^4 + 13D^2 + 36)x = -60\sin t.$$

The associated homogeneous equation has characteristic equation $\left(r^2+4\right)\left(r^2+9\right)=0$ and complementary function $x_c = a_1\cos 2t + a_2\sin 2t + b_1\cos 3t + b_2\sin 3t$. Upon substituting the trial solution $x_p = A\sin t$ we find that $A = \frac{5}{2}$. Thus the general solution for *x* is

$$x(t) = a_1\cos 2t + a_2\sin 2t + b_1\cos 3t + b_2\sin 3t + \frac{5}{2}\sin t.$$

We find similarly that

$$y(t) = c_1\cos 2t + c_2\sin 2t + d_1\cos 3t + d_2\sin 3t + \frac{1}{2}\cos t.$$

When we substitute these expressions into either of the original differential equations we find that $c_1 = -\frac{4}{13}a_2$, $c_2 = \frac{4}{13}a_1$, $d_1 = -\frac{1}{3}b_2$, and $d_2 = \frac{1}{3}b_1$. Therefore

$$y(t) = \frac{4}{13}\left(-a_2\cos 2t + a_1\sin 2t\right) + \frac{1}{3}\left(-b_2\cos 3t + b_1\sin 3t\right) + \frac{1}{2}\cos t.$$

17. If we write the given equations in the operational form

$$\begin{aligned}\left(D^2-3D-2\right)x &+ \left(D^2-D+2\right)y = 0,\\ \left(2D^2-9D-4\right)x &+ \left(3D^2-2D+6\right)y = 0,\end{aligned}$$

we see (thinking of the operational determinant) that *x* satisfies a homogeneous fourth-order equation with characteristic equation

$$\begin{aligned}\left(r^2-3r-2\right)\left(3r^2-2r+6\right)-\left(r^2-r+2\right)\left(2r^2-9r-4\right) &= r^4-3r^2-4\\ &= \left(r^2+1\right)\left(r^2-4\right)\\ &= \left(r^2+1\right)(r+2)(r-2)\\ &= 0.\end{aligned}$$

Hence the general solution for *x* is

$$x(t) = a_1\cos t + a_2\sin t + b_1e^{-2t} + b_2e^{2t};$$

similarly, the general solution for *y* is

$$y(t) = c_1\cos t + c_2\sin t + d_1e^{-2t} + d_2e^{2t}.$$

To determine the relations between the arbitrary constants in these two general solutions, we substitute them in the first of the original differential equations and get

$$\left(-a_1\cos t - a_2\sin t + 4b_1e^{-2t} + 4b_2e^{2t}\right) + \left(-c_1\cos t - c_2\sin t + 4d_1e^{-2t} + 4d_2e^{2t}\right) +$$
$$\left(3a_1\sin t - 3a_2\cos t + 6b_1e^{-2t} - 6b_2e^{2t}\right) + \left(c_1\sin t - c_2\cos t + 2d_1e^{-2t} - 2d_2e^{2t}\right) +$$
$$\left(-2a_1\cos t - 2a_2\sin t - 2b_1e^{-2t} - 2b_2e^{2t}\right) + \left(+2c_1\cos t + 2c_2\sin t + 2d_1e^{-2t} + 2d_2e^{2t}\right) = 0.$$

If we collect coefficients of the trigonometric and exponential terms we get the equations

$$\begin{aligned} -3a_1 - 3a_2 + c_1 - c_2 &= 0, \\ 3a_1 - 3a_2 + c_1 + c_2 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 8b_1 + 8d_1 &= 0, \\ -4b_2 + 4d_2 &= 0. \end{aligned}$$

The first two of these equations imply that $c_1 = -3a_2$ and $c_2 = 3a_1$, while the latter two give $d_1 = -b_1$ and $d_2 = b_2$. We therefore see finally that

$$y(t) = -3a_2 \cos t + 3a_1 \sin t - b_1 e^{-2t} + b_2 e^{2t}.$$

18. Adding the first and third equations gives $x' + z' = 0$. Then differentiation of the first equation gives

$$x'' = \cancel{x'} + 2y' + \cancel{z'} = 2y'$$

Then differentiating the second equation twice and substituting $2y'$ for x'' gives

$$y''' = 6x'' - y'' = 12y' - y'',$$

or $y''' + y'' - 12y' = 0$, with characteristic equation $r^3 + r^2 - 12r = 0$ and characteristic roots $r = 0, 3, -4$, so that

$$y(t) = c_0 + c_1 e^{3t} + c_2 e^{-4t}.$$

Then the second given equation implies that

$$x(t) = \frac{1}{6}(y + y') = \frac{1}{6}\left(c_0 + 4c_1 e^{3t} - 3c_2 e^{-4t}\right).$$

Finally, the first given equation completes the solution:

$$z(t) = x' - x - 2y = \frac{1}{6}\left(-13c_0 - 4c_1 e^{3t} + 3c_2 e^{-4t}\right).$$

19. The operational determinant of the given system is

$$L = \begin{vmatrix} D-4 & 2 & 0 \\ 4 & D-4 & 2 \\ 0 & 4 & D-4 \end{vmatrix} = D^3 - 12D^2 + 32D,$$

so x, y, and z all satisfy a third-order homogeneous linear differential equation with characteristic equation $r^3 - 12r^2 + 32r = r(r-4)(r-8) = 0$. The corresponding general solutions are

$$x(t) = a_1 + a_2 e^{4t} + a_3 e^{8t}, \quad y(t) = b_1 + b_2 e^{4t} + b_3 e^{8t}, \quad z(t) = c_1 + c_2 e^{4t} + c_3 e^{8t}.$$

If we substitute $x(t)$ and $y(t)$ in the first differential equation $x' = 4x - 2y$ and collect coefficients of like terms, we find quickly that $b_1 = 2a_1$, $b_2 = 0$, and $b_3 = -2a_3$. Similarly, we find by substitution in the other two equations that $c_1 = 2a_1$, $c_2 = -2a_2$, and $c_3 = 2a_3$. Thus y and z are given by

$$y(t) = 2a_1 - 2a_3e^{8t}, \quad z(t) = 2a_1 - 2a_2e^{4t} + 2a_3e^{8t}.$$

20. The operational determinant of the given system is

$$L = D^3 - 3D - 2 = (D+1)^2(D-2),$$

and we find that

$$Lx = Ly = Lz = 0.$$

Hence

$$x = a_1e^{2t} + a_2e^{-t} + a_3te^{-t},$$
$$y = b_1e^{2t} + b_2e^{-t} + b_3te^{-t},$$
$$z = c_1e^{2t} + c_2e^{-t} + c_3te^{-t}.$$

When we substitute these expressions in the three differential equations and compare coefficients of e^{2t}, we find that $a_1 = b_1 = c_1$. When we compare coefficients of te^{-t} we find that $a_3 + b_3 + c_3 = 0$. Comparison of coefficients of e^{-t} yields

$$a_2 + b_2 + c_2 = a_3 - 1 = b_3 = c_3.$$

It follows that $a_3 = \frac{2}{3}$ and $b_3 = c_3 = -\frac{1}{3}$. If a_2 and b_2 are chosen arbitrarily, then $c_2 = -a_2 - b_2 - \frac{1}{3}$. Hence the general solution is

$$x = a_1e^{2t} + a_2e^{-t} + \frac{2}{3}te^{-t},$$
$$y = a_1e^{2t} + b_2e^{-t} - \frac{1}{3}te^{-t},$$
$$z = a_1e^{2t} - \left(a_2 + b_2 + \frac{1}{3}\right)e^{-t} - \frac{1}{3}te^{-t}.$$

21. $L_1L_2 = L_2L_1$ because both sides simplify to the same thing upon multiplying out and collecting terms in the usual fashion of polynomial algebra. This "works" because different powers of *D* commute—that is, $D^iD^j = D^jD^i$ because $D^i\left(D^jx\right) = D^{i+j}x = D^j\left(D^ix\right)$.

22. The following calculations show that $L_1L_2x \neq L_2L_1x$ in general:

$$L_1(L_2x) = (tD+1)(Dx+tx) = tD(Dx+tx) + (Dx+tx)$$
$$= t\left(D^2x + tDx + x\right) + (Dx+tx) = tD^2x + t^2Dx + Dx + 2tx,$$

whereas

$$\begin{aligned} L_2(L_1x) &= (D+t)(tDx+x) = D(tDx+x) + t(tDx+x) \\ &= (tD^2x + 2Dx) + (t^2Dx + tx) = tD^2x + t^2Dx + 2Dx + tx. \end{aligned}$$

23. Subtraction of the two equations yields $x + y = e^{-2t} - e^{-3t}$. We then verify readily that any two differentiable functions $x(t)$ and $y(t)$ satisfying this condition will constitute a solution of the given system, which thus has infinitely many solutions.

24. Subtraction of one equation from the other yields $x + y = t^2 - t$. But then

$$(D+2)x + (D+2)y = D(x+y) + 2(x+y) = (2t-1) + 2(t^2 - t) = 2t^2 - 1 \neq t.$$

Thus the given system has no solution.

25. Infinitely many solutions, because any solution of the second equation also satisfies the first equation (because it is $D+2$ times the second one).

26. Subtraction of the second equation from the first one gives $x = e^{-t}$. Then substitution in the second equation yields $D^2y = 0$, so $y = b_1t + b_2$. Thus there are *two* arbitrary constants.

27. Subtraction of the second equation from the first one gives $x + y = e^{-t}$. Then substitution in the second equation yields

$$x(t) = D^2(x+y) = e^{-t}.$$

It follows that $y(t) \equiv 0$, so there are *no* arbitrary constants.

28. Differentiation of the difference of the two given equations yields

$$(D^2 + D)x + D^2y = -2e^{-t},$$

which contradicts the first equation. Hence the system has *no* solution.

29. Addition of the two given equations yields $D^2x = e^{-t}$, so $x(t) = e^{-t} + a_1t + a_2$. Then the second equation gives $D^2y = a_1t + a_2$, so

$$y(t) = \frac{1}{6}a_1t^3 + \frac{1}{2}a_2t^2 + a_3t + a_4.$$

Thus there are *four* arbitrary constants.

30. Substitution of $y = 20x' + 6x$ and $y' = 20x'' + 6x'$ —from the first equation—into the second equation yields the second-order equation $100x'' + 45x' + 3x = 0$ with general solution $x(t) = c_1e^{r_1t} + c_2e^{r_2t}$, where

$$r_1 = \frac{-9+\sqrt{33}}{40}, \quad r_2 = \frac{-9-\sqrt{33}}{40}.$$

Substitution in $y = 20x' + 6x$ now yields

$$y(t) = \frac{1}{2}\left(3+\sqrt{33}\right)c_1 e^{r_1 t} + \frac{1}{2}\left(3-\sqrt{33}\right)c_2 e^{r_2 t}.$$

Imposition of the initial conditions $x(0) = 50$, $y(0) = 100$ now gives the equations

$$c_1 + c_2 = 50, \quad \left(3+\sqrt{33}\right)c_1 + \left(3-\sqrt{33}\right)c_2 = 200$$

with solution $c_1 = \frac{25}{\sqrt{33}}\left(1+\sqrt{33}\right)$, $c_2 = \frac{25}{\sqrt{33}}\left(-1+\sqrt{33}\right)$. These coefficients give the desired particular solution

$$x(t) = \frac{25}{\sqrt{33}}\left[\left(1+\sqrt{33}\right)e^{r_1 t} + \left(-1+\sqrt{33}\right)e^{r_2 t}\right],$$
$$y(t) = \frac{50}{11}\left[\left(11+3\sqrt{33}\right)e^{r_1 t} + \left(11-3\sqrt{33}\right)e^{r_2 t}\right].$$

31. Substitution of $I_2 = \frac{1}{25}\left(I_1' + 25I_1 - 50\right)$ and $I_2' = \frac{1}{25}\left(I_1'' + 25I_1'\right)$ —from the first equation—into the second equation yields the second-order equation $3I_1'' + 30I_1' + 125I_1 = 250$ with general solution

$$I_1(t) = 2 + e^{-5t}\left[c_1\cos\left(\frac{5\sqrt{6}}{3}t\right) + c_2\sin\left(\frac{5\sqrt{6}}{3}t\right)\right].$$

Substitution in $I_2 = \frac{1}{25}\left(I_1' + 25I_1 - 50\right)$ now yields

$$I_2(t) = \frac{1}{15}e^{-5t}\left[\left(12c_1 + \sqrt{6}c_2\right)\cos\left(\frac{5\sqrt{6}}{3}t\right) + \left(12c_2 - \sqrt{6}c_1\right)\sin\left(\frac{5\sqrt{6}}{3}t\right)\right].$$

Imposition of the initial conditions $I_1(0) = 0$, $I_2(0) = 0$ now gives the equations

$$c_1 + 2 = 0, \quad \frac{4}{5}c_1 + \frac{\sqrt{6}}{15}c_2 = 0$$

with solution $c_1 = -2$, $c_2 = 4\sqrt{6}$. These coefficients give the desired particular solution

$$I_1(t) = 2 + e^{-5t}\left[-2\cos\left(\frac{5\sqrt{6}}{3}t\right) + 4\sqrt{6}\sin\left(\frac{5\sqrt{6}}{3}t\right)\right],$$
$$I_2(t) = \frac{20}{\sqrt{6}}e^{-5t}\sin\left(\frac{5\sqrt{6}}{3}t\right).$$

32. To solve the system

$$2(I_1' - I_2') + 50I_1 = 100\sin 60t,$$
$$2(I_2' - I_1') + 25I_2 = 0$$

we first note from the second equation that $2(I_1' - I_2') = 25I_2$, so the first equation then tells us that $25I_2 + 50I_1 = 100\sin 60t$, hence $I_2 = -2I_1 + 4\sin 60t$, and $I_2' = -2I_1' + 240\cos 60t$. Substitution of these into the second equation gives the first-order equation

$$6I_1' + 50I_1 = 480\cos 60t + 100\sin 60t$$

with complementary function $I_{1c} = Ce^{-25t/3}$. Substitution of the trial solution $I_{1p} = A\cos 60t + B\sin 60t$ yields $A = -\dfrac{120}{1321}$, $B = \dfrac{1778}{1321}$. Finally, the initial condition $I_1(0) = 0$ gives $C = A$. Consequently we find that

$$I_1(t) = \frac{1}{1321}\left(120e^{-25t/3} - 120\cos 60t + 1778\sin 60t\right),$$
$$I_2(t) = \frac{1}{1321}\left(-240e^{-25t/3} + 240\cos 60t + 1728\sin 60t\right).$$

(Yes, one coefficient of $\sin 60t$ is 1778 and the other is 1728.)

33. To solve the system

$$I_1' = -20(I_1 - I_2), \quad I_2' = 40(I_1 - I_2)$$

we first note that $I_2' = -2I_1'$, so $I_2 = -2I_1 + K$. Then $K = 2I_1(0) + I_2(0) = 2\cdot 2 + 0 = 4$, so $I_2' = -2I_1'$, so $I_2 = -2I_1 + 4$. Substitution of this into the first equation gives the simple first-order linear equation $I_1' + 60I_1 = 80$ with general solution $I_1(t) = \dfrac{4}{3} + ce^{-60t}$. The initial condition $I_1(0) = 2$ gives $c = \dfrac{2}{3}$, so

$$I_1(t) = \frac{2}{3}\left(2 + e^{-60t}\right), \quad I_2(t) = \frac{4}{3}\left(1 - e^{-60t}\right).$$

34. The operational determinant of the system

$$10x_1' = -x_1 + x_3, \quad 10x_2' = x_1 - x_2, \quad 10x_3' = x_2 - x_3$$

is

$$L = \begin{vmatrix} 10D+1 & 0 & -1 \\ -1 & 10D+1 & 0 \\ 0 & -1 & 10D+1 \end{vmatrix} = 1000D^3 + 300D^2 + 30D,$$

so x_1, x_2, and x_3 all satisfy a third-order homogeneous linear differential equation with characteristic equation

$$1000r^3 + 300r^2 + 30r = 10r\left(100r^2 + 30r + 3\right) = 0$$

and characteristic roots $r = 0, \frac{1}{20}\left(-3 \pm i\sqrt{3}\right)$. The corresponding general solutions are

$$x_1(t) = a_1 + e^{-3t/20}\left[b_1 \cos\left(\frac{\sqrt{3}}{20}t\right) + c_1 \sin\left(\frac{\sqrt{3}}{20}t\right)\right],$$

$$x_2(t) = a_2 + e^{-3t/20}\left[b_2 \cos\left(\frac{\sqrt{3}}{20}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{20}t\right)\right],$$

$$x_3(t) = a_3 + e^{-3t/20}\left[b_3 \cos\left(\frac{\sqrt{3}}{20}t\right) + c_3 \sin\left(\frac{\sqrt{3}}{20}t\right)\right].$$

Substituting in the original differential equations, we see that the constant terms are all equal: $a_1 = a_2 = a_3 = a$ (say). Then the initial conditions $x_1(0) = 100$, $x_2(0) = x_3(0) = 0$ imply that $b_1 = 100 - a$, $b_2 = b_3 = -a$. After these substitutions, collection of coefficients gives the equations

$$\frac{3}{2}a + \frac{\sqrt{3}}{2}c_1 - 50 = 0, \quad \frac{\sqrt{3}}{2}a - \frac{1}{2}c_1 - c_3 - 50\sqrt{3} = 0,$$

$$\frac{3}{2}a + \frac{\sqrt{3}}{2}c_2 - 100 = 0, \quad \frac{\sqrt{3}}{2}a - c_1 - \frac{1}{2}c_2 = 0$$

that we solve for $a = \frac{100}{3}$, $c_1 = 0$, $c_2 = \frac{100}{\sqrt{3}}$, and $c_3 = -\frac{100}{\sqrt{3}}$. The resulting solution of the original system is given by

$$x_1(t) = \frac{100}{3}\left[1 + 2e^{-3t/20}\cos\left(\frac{\sqrt{3}}{20}t\right)\right],$$

$$x_2(t) = \frac{100}{3}\left\{1 + e^{-3t/20}\left[-\cos\left(\frac{\sqrt{3}}{20}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{20}t\right)\right]\right\},$$

$$x_3(t) = \frac{100}{3}\left\{1 + e^{-3t/20}\left[-\cos\left(\frac{\sqrt{3}}{20}t\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}}{20}t\right)\right]\right\}.$$

35. The two given equations yield

$$mx^{(3)} = qBy'' = -\frac{q^2B^2}{m}x',$$

so $x^{(3)} + \omega^2 x' = 0$. The general solution is

$$x(t) = A\cos\omega t + B\sin\omega t + C.$$

Now $x'(0) = 0$ implies $B = 0$, and then $x(0) = r_0$ gives $A + C = r_0$. Next,

$$\omega y' = x'' = -A\omega^2\cos\omega t,$$

so $y'(0) = -\omega r_0$ implies that $A = r_0$, and hence that $C = 0$. It now follows readily that the trajectory is the circle

$$x(t) = r_0\cos\omega t, \quad y(t) = -r_0\sin\omega t.$$

36. With $\omega = \dfrac{qB}{m}$ our differential equations are

$$x'' = \omega y' + \frac{qE}{m}, \quad y'' = -\omega x'.$$

Elimination gives

$$x^{(4)} + \omega^2 x'' = y^{(4)} + \omega^2 y'' = 0,$$

so

$$\begin{aligned} x &= a_1 + b_1 t + c_1\cos\omega t + d_1\sin\omega t, \\ y &= a_2 + b_2 t + c_2\cos\omega t + d_2\sin\omega t. \end{aligned}$$

The initial conditions $x(0) = x'(0) = 0 = y(0) = y'(0)$ yield $c_1 = -a_1$, $b_1 = -\omega d_1$, $c_2 = -a_2$, and $b_2 = -\omega d_2$. Then substitution in $y'' = -\omega x'$ yields $d_1 = b_1 = a_2 = c_2 = 0$ and $d_2 = a_1$. Finally, substitution in $x'' = \omega y' + \dfrac{qE}{m}$ yields $a = a_1 = \dfrac{E}{\omega B}$, so the solution is

$$x(t) = a(1 - \cos\omega t), \quad y(t) = -a(\omega t - \sin\omega t).$$

37. **(a)** If we set $m_1 = 2$, $m_2 = \dfrac{1}{2}$, $k_1 = 75$, and $k_2 = 25$ in Eqs. (3) of Section 4.1, we get the system

$$2x'' = -100x + 25y, \quad \frac{1}{2}y'' = 25x - 25y$$

with operational determinant $D^4 + 100D^2 + 1875 = (D^2 + 25)(D^2 + 75)$. Hence the general form of the solution is

$$\begin{aligned} x(t) &= a_1\cos 5t + a_2\sin 5t + b_1\cos 5\sqrt{3}t + b_2\sin 5\sqrt{3}t, \\ y(t) &= c_1\cos 5t + c_2\sin 5t + d_1\cos 5\sqrt{3}t + d_2\sin 5\sqrt{3}t. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = 2a_1$ and $c_2 = 2a_2$, and $d_1 = -2b_1$ and $d_2 = -2b_2$. This gives

$$x(t) = a_1 \cos 5t + a_2 \sin 5t + b_1 \cos 5\sqrt{3}t + b_2 \sin 5\sqrt{3}t,$$
$$y(t) = 2a_1 \cos 5t + 2a_2 \sin 5t - 2b_1 \cos 5\sqrt{3}t - 2b_2 \sin 5\sqrt{3}t.$$

(b) In the natural mode with frequency $\omega_1 = 5$ the masses move in the same direction, while in the natural mode with frequency $\omega_2 = 5\sqrt{3}$ they move in opposite directions. In each case the amplitude of the motion of m_2 is twice that of m_1.

38. Looking at Fig. 4.2.6 in the text, we see that the first spring is stretched by x, the second spring is stretched by $y - x$, and the third spring is compressed by y. Hence Newton's second law gives $m_1 x'' = -k_1(x) + k_2(y - x)$ and $m_2 y'' = -k_2(y - x) - k_3(y)$.

39. The system has operational determinant $8D^4 + 40D^2 + 32 = 8(D^2 + 1)(D^2 + 4)$. Hence the general form of the solution is

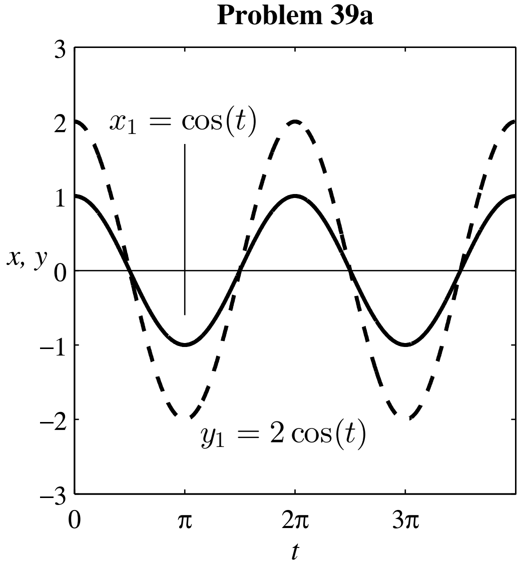
$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t,$$
$$y(t) = c_1 \cos t + c_2 \sin t + d_1 \cos 2t + d_2 \sin 2t.$$

Upon substitution in either differential equation we see that $c_1 = 2a_1$ and $c_2 = 2a_2$, and $d_1 = -b_1$ and $d_2 = -b_2$. This gives

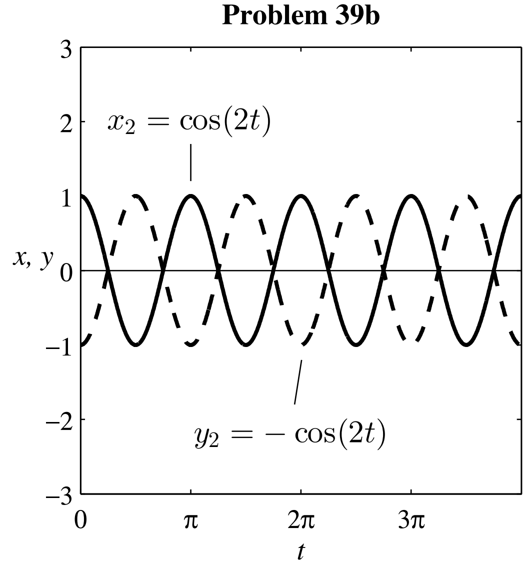
$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t,$$
$$y(t) = 2a_1 \cos t + 2a_2 \sin t - b_1 \cos 2t - b_2 \sin 2t.$$

In the natural mode with frequency $\omega_1 = 1$ the masses move in the same direction, with the amplitude of motion of the second mass twice that of the first mass (figure **a**). In the natural mode with frequency $\omega_2 = 2$ they move in opposite directions with the same amplitude of motion (figure **b**).

Problem 39a



Problem 39b



40. The system has operational determinant $2D^4 + 250D^2 + 5000 = 2(D^2 + 25)(D^2 + 100)$. Hence the general form of the solution is

$$x(t) = a_1 \cos 5t + a_2 \sin 5t + b_1 \cos 10t + b_2 \sin 10t,$$
$$y(t) = c_1 \cos 5t + c_2 \sin 5t + d_1 \cos 10t + d_2 \sin 10t.$$

Upon substitution in either differential equation we see that $c_1 = 2a_1$ and $c_2 = 2a_2$, and $d_1 = -b_1$ and $d_2 = -b_2$. This gives

$$x(t) = a_1 \cos 5t + a_2 \sin 5t + b_1 \cos 10t + b_2 \sin 10t,$$
$$y(t) = 2a_1 \cos 5t + 2a_2 \sin 5t - b_1 \cos 10t - b_2 \sin 10t.$$

In the natural mode with frequency $\omega_1 = 5$ the masses move in the same direction, with the amplitude of motion of the second mass twice that of the first mass. In the natural mode with frequency $\omega_2 = 10$ they move in opposite directions with the same amplitude of motion.

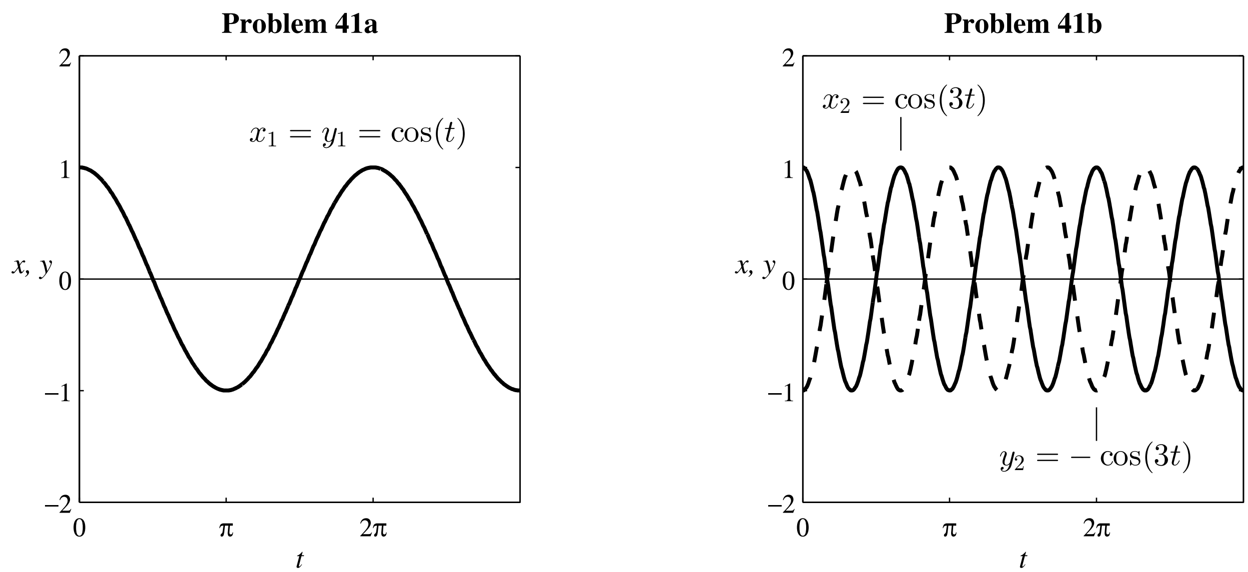
41. The system has operational determinant $D^4 + 10D^2 + 9 = (D^2 + 1)(D^2 + 9)$. Hence the general form of the solution is

$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t,$$
$$y(t) = c_1 \cos t + c_2 \sin t + d_1 \cos 3t + d_2 \sin 3t.$$

Upon substitution in either differential equation we see that $c_1 = a_1$ and $c_2 = a_2$, and $d_1 = -b_1$ and $d_2 = -b_2$. This gives

$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t,$$
$$y(t) = a_1 \cos t + a_2 \sin t - b_1 \cos 3t - b_2 \sin 3t.$$

In the natural mode with frequency $\omega_1 = 1$ the masses move in the same direction (figure **a**), while in the natural mode with frequency $\omega_2 = 3$ they move in opposite directions (figure **b**). In each case the amplitudes of motion of the two masses are equal.



42. The system has operational determinant $2D^4 + 10D^2 + 8 = 2(D^2+1)(D^2+4)$. Hence the general form of the solution is

$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t,$$
$$y(t) = c_1 \cos t + c_2 \sin t + d_1 \cos 2t + d_2 \sin 2t.$$

Upon substitution in either differential equation we see that $c_1 = a_1$ and $c_2 = a_2$, and $d_1 = -\frac{1}{2}b_1$ and $d_2 = -\frac{1}{2}b_2$. This gives

$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t,$$
$$y(t) = a_1 \cos t + a_2 \sin t - \frac{1}{2}b_1 \cos 2t - \frac{1}{2}b_2 \sin 2t.$$

In the natural mode with frequency $\omega_1 = 1$ the two masses move in the same direction with equal amplitudes of oscillation. In the natural mode with frequency $\omega_2 = 2$ the two masses move in opposite directions with the amplitude of m_2 being half that of m_1.

43. The system has operational determinant $D^4 + 6D^2 + 5 = (D^2+1)(D^2+5)$. Hence the general form of the solution is

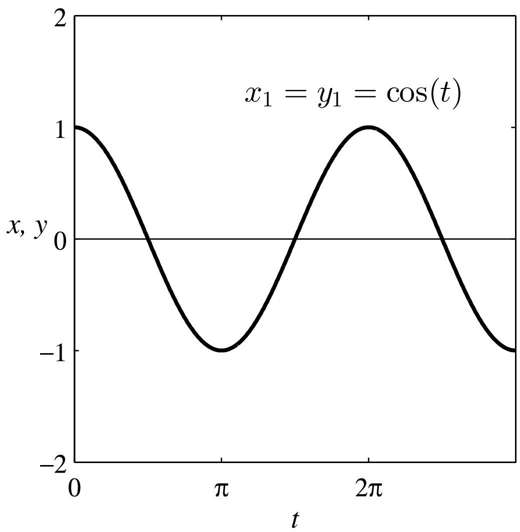
$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos\left(\sqrt{5}t\right) + b_2 \sin\left(\sqrt{5}t\right),$$
$$y(t) = c_1 \cos t + c_2 \sin t + d_1 \cos\left(\sqrt{5}t\right) + d_2 \sin\left(\sqrt{5}t\right).$$

Upon substitution in either differential equation we see that $c_1 = a_1$ and $c_2 = a_2$, and $d_1 = -b_1$ and $d_2 = -b_2$. This gives

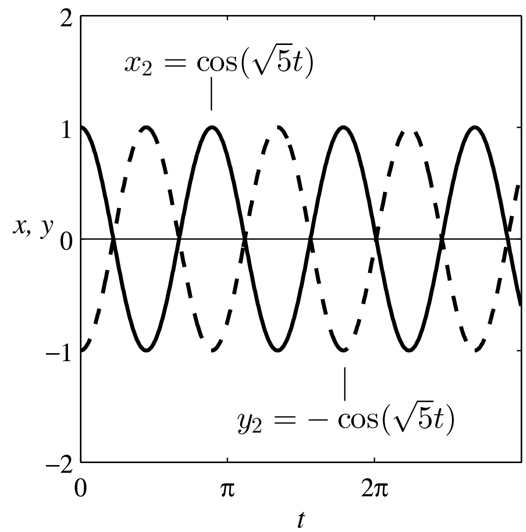
$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos\left(\sqrt{5}t\right) + b_2 \sin\left(\sqrt{5}t\right),$$
$$y(t) = a_1 \cos t + a_2 \sin t - b_1 \cos\left(\sqrt{5}t\right) - b_2 \sin\left(\sqrt{5}t\right).$$

In the natural mode with frequency $\omega_1 = 1$ the masses move in the same direction (figure **a**), while in the natural mode with frequency $\omega_2 = \sqrt{5}$ they move in opposite directions (figure **b**). In each case the amplitudes of motion of the two masses are equal.

Problem 43a



Problem 43b



44. The system has operational determinant $D^4 + 6D^2 + 8 = \left(D^2 + 2\right)\left(D^2 + 4\right)$. Hence the general form of the solution is

$$x(t) = a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + b_1 \cos 2t + b_2 \sin 2t,$$
$$y(t) = c_1 \cos t\sqrt{2} + c_2 \sin t\sqrt{2} + d_1 \cos 2t + d_2 \sin 2t.$$

Upon substitution in either differential equation we see that $c_1 = a_1$ and $c_2 = a_2$, and $d_1 = -b_1$ and $d_2 = -b_2$. This gives

$$x(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) + b_1 \cos 2t + b_2 \sin 2t,$$
$$y(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) - b_1 \cos 2t - b_2 \sin 2t.$$

In the natural mode with frequency $\omega_1 = \sqrt{2}$ the two masses move in the same direction; in the natural mode with frequency $\omega_2 = 2$ they move in opposite directions. In each natural mode their amplitudes of oscillation are equal.

45. The system has operational determinant $2D^4 + 20D^2 + 32 = 2\left(D^2 + 2\right)\left(D^2 + 8\right)$. Hence the general form of the solution is

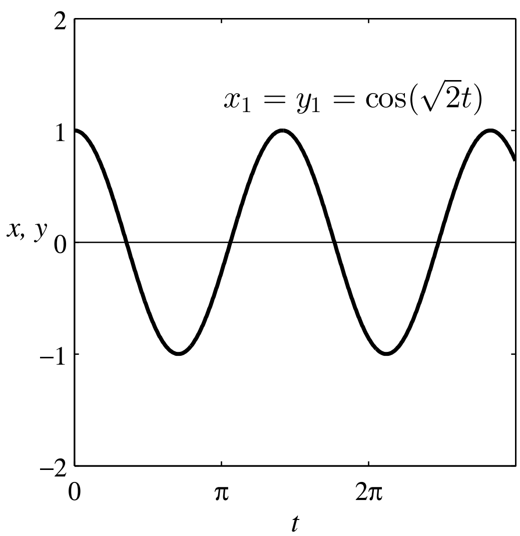
$$x(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) + b_1 \cos\left(\sqrt{8}t\right) + b_2 \sin\left(\sqrt{8}t\right),$$
$$y(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right) + d_1 \cos\left(\sqrt{8}t\right) + d_2 \sin\left(\sqrt{8}t\right).$$

Upon substitution in either differential equation we see that $c_1 = a_1$ and $c_2 = a_2$, and $d_1 = -\frac{1}{2}b_1$ and $d_2 = -\frac{1}{2}b_2$. This gives

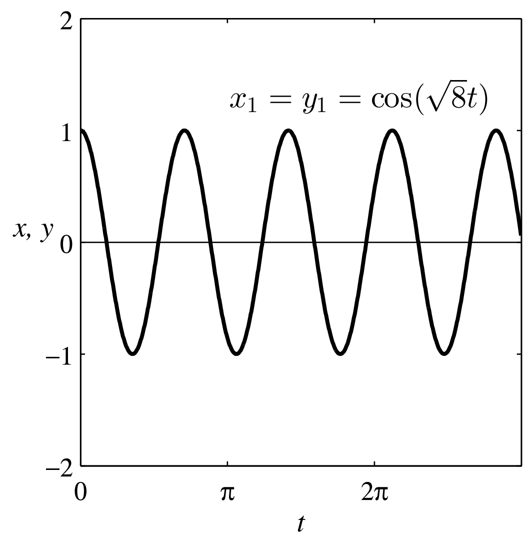
$$x(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) + b_1 \cos\left(\sqrt{8}t\right) + b_2 \sin\left(\sqrt{8}t\right),$$
$$y(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) - \frac{1}{2}b_1 \cos\left(\sqrt{8}t\right) - \frac{1}{2}b_2 \sin\left(\sqrt{8}t\right).$$

In the natural mode with frequency $\omega_1 = \sqrt{2}$ the two masses move in the same direction with equal amplitudes of oscillation (figure **a**). In the natural mode with frequency $\omega_2 = \sqrt{8} = 2\sqrt{2}$ the two masses move in opposite directions with the amplitude of m_2 being half that of m_1 (figure **b**).

Problem 45a



Problem 45b



46. The system has operational determinant $D^4 + 20D^2 + 64 = \left(D^2 + 4\right)\left(D^2 + 16\right)$. Hence the general form of the solution is

$$x(t) = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t,$$
$$y(t) = c_1 \cos 2t + c_2 \sin 2t + d_1 \cos 4t + d_2 \sin 4t.$$

Upon substitution in either differential equation we see that $c_1 = a_1$ and $c_2 = a_2$, and $d_1 = -b_1$ and $d_2 = -b_2$. This gives

$$x(t) = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t,$$
$$y(t) = a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t.$$

In the natural mode with frequency $\omega_1 = 2$ the masses move in the same direction with equal amplitudes of motion. In the natural mode with frequency $\omega_2 = 4$ they move in opposite directions with the same amplitude of motion.

47. **(a)** Looking at Fig. 4.2.7 in the text, we see that the first spring is stretched by x, the second spring is stretched by $y-x$, the third spring is stretched by $z-y$, and the fourth spring is compressed by z. Hence Newton's second law gives $mx'' = -k(x)+k(y-x)$, $my'' = -k(y-x)+k(z-y)$, and $mz'' = -k(z-y)-k(z)$.

(b) The operational determinant is

$$(D^2+2)\left[(D^2+2)^2-1\right]+\left[-(D^2+2)\right]=(D^2+2)\left[(D^2+2)^2-2\right],$$

and the characteristic equation $(r^2+2)\left[(r^2+2)^2-2\right]=0$ has roots $\pm i\sqrt{2}$ and $\pm i\sqrt{2\pm\sqrt{2}}$.

48. The given system has operational determinant $D^4+10D^2+9=(D^2+1)(D^2+9)$. Hence the general form of the solution is

$$x(t) = a_1\cos t + a_2\sin t + b_1\cos 3t + b_2\sin 3t,$$
$$y(t) = c_1\cos t + c_2\sin t + d_1\cos 3t + d_2\sin 3t.$$

Upon substitution in either differential equation we see that $c_1 = -a_2$ and $c_2 = a_1$, and $d_1 = b_2$ and $d_2 = -b_1$. This gives

$$x(t) = a_1\cos t + a_2\sin t + b_1\cos 3t + b_2\sin 3t,$$
$$y(t) = -a_2\cos t + a_1\sin t + b_2\cos 3t - b_1\sin 3t.$$

When we impose the initial conditions $x(0)=4$ and $y(0)=x'(0)=y'(0)=0$ we find that $a_1 = 3$, $b_1 = 1$, and $a_2 = b_2 = 0$.

SECTION 4.3

NUMERICAL METHODS FOR SYSTEMS

In Problems 1-8 we first write the given system in the form $x' = f(t,x,y)$ $y' = g(t,x,y)$. Then we use the template

$$h = 0.1; \qquad t_1 = t_0 + h$$
$$x_1 = x_0 + hf(t_0,x_0,y_0); \quad y_1 = y_0 + hg(t_0,x_0,y_0)$$
$$x_2 = x_1 + hf(t_1,x_1,y_1); \quad y_2 = y_1 + hg(t_1,x_1,y_1)$$

(with the given values of t_0, x_0, and y_0) to calculate the Euler approximations $x_1 \approx x(0.1)$ and $y_1 \approx y(0.1)$, and $x_2 \approx x(0.2)$ and $y_2 \approx y(0.2)$, in part (a). We give these approximations and the actual values $x_{\text{act}} = x(0.2)$, $y_{\text{act}} = y(0.2)$ in tabular form. We use the template

$$h = 0.2; \qquad t_1 = t_0 + h$$
$$u_1 = x_0 + hf(t_0, x_0, y_0); \qquad v_1 = y_0 + hg(t_0, x_0, y_0)$$
$$x_1 = x_0 + \frac{1}{2}h\left[f(t_0, x_0, y_0) + f(t_1, u_1, v_1)\right] \quad y_1 = y_0 + \frac{1}{2}h\left[g(t_0, x_0, y_0) + g(t_1, u_1, v_1)\right]$$

to calculate the improved Euler approximations $u_1 \approx x(0.2)$ and $u_1 \approx y(0.2)$, and $x_1 \approx x(0.2)$ and $y_1 \approx y(0.2)$, in part (b). We give these approximations and the actual values $x_{\text{act}} = x(0.2)$, $y_{\text{act}} = y(0.2)$ in tabular form. We use the template

$$h = 0.2;$$
$$F_1 = f(t_0, x_0, y_0); \qquad G_1 = g(t_0, x_0, y_0);$$
$$F_2 = f\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}F_1, y_0 + \frac{1}{2}hG_1\right); \quad G_2 = g\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}F_1, y_0 + \frac{h}{2}G_1\right);$$
$$F_3 = f\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}F_2, y_0 + \frac{h}{2}G_2\right); \quad G_3 = g\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}F_2, y_0 + \frac{h}{2}G_2\right);$$
$$F_4 = f(t_0 + h, x_0 + hF_3, y_0 + hG_3); \qquad G_4 = g(t_0 + h, x_0 + hF_3, y_0 + hG_3);$$
$$x_1 = x_0 + \frac{h}{6}(F_1 + 2F_2 + 2F_3 + F_4); \qquad y_1 = y_0 + \frac{h}{6}(G_1 + 2G_2 + 2G_3 + G_4)$$

to calculate the intermediate slopes and Runge-Kutta approximations $x_1 \approx x(0.2)$ and $y_1 \approx y(0.2)$ for part (c). Again, we give the results in tabular form.

1. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.4	2.2	0.88	2.5	1.0034	2.6408

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
0.8	2.4	0.96	2.6	1.0034	2.6408

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
4	2	4.8	3	5.08	3.26	6.32	4.684
x_1	y_1	x_{act}	y_{act}				
1.0027	2.6401	1.0034	2.6408				

2. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	–0.9	0.81	–0.81	0.8187	–0.8187

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
0.8	–0.8	0.82	–0.82	0.8187	–0.8187

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–1	1	–0.9	0.9	–0.91	0.91	–0.818	0.818
x_1	y_1	x_{act}	y_{act}				
0.8187	–0.8187	0.8187	–0.8187				

3. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
1.7	1.5	2.81	2.31	3.6775	2.9628

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
2.4	2	3.22	2.62	3.6775	2.9628

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
7	5	11.1	8.1	13.57	9.95	23.102	17.122
x_1	y_1	x_{act}	y_{act}				
3.6481	2.9407	3.6775	2.9628				

4. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
1.9	–0.6	3.31	–1.62	4.2427	-2.4205

(b)

u_1	..	x_1	y_1	x_{act}	y_{act}
2.8	–1.2	3.82	–2.04	4.2427	-2.4205

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
9	–6	14.1	–10.2	16.59	–12.42	26.442	–20.94
x_1	y_1	x_{act}	y_{act}				
4.2274	–2.4060	4.2427	-2.4205				

5. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	3.2	–0.52	2.92	-0.5793	2.4488

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
–0.2	3.4	–0.84	2.44	-0.5793	2.4488

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–11	2	–14.2	–2.8	–12.44	–3.12	–12.856	–6.704
x_1	y_1	x_{act}	y_{act}				
–0.5712	2.4485	-0.5793	2.4488				

6. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
–0.8	4.4	–1.76	4.68	-1.9025	4.4999

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
–1.6	4.8	–1.92	4.56	-1.9025	4.4999

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–8	4	–9.6	2.8	–9.52	2.36	–10.848	0.664
x_1	y_1	x_{act}	y_{act}				
–1.9029	4.4995	-1.9025	4.4999				

7. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
2.5	1.3	3.12	1.68	3.2820	1.7902

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
3	1.6	3.24	1.76	3.2820	1.7902

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
5	3	6.2	3.8	6.48	4	8.088	5.096
x_1	y_1	x_{act}	y_{act}				
3.2816	1.7899	3.2820	1.7902				

8. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	–0.9	2.16	–0.63	2.5270	-0.3889

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
1.8	–0.8	2.52	–0.46	2.5270	-0.3889

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
9	1	12.6	2.7	12.87	3.25	16.02	5.498
x_1	y_1	x_{act}	y_{act}				
2.5320	–0.3867	2.5270	-0.3889				

In Problems 9-11 we use the same Runge-Kutta template as in part (c) of Problems 1–8 above, and give both the Runge-Kutta approximate values with step sizes $h = 0.1$ and $h = 0.05$, and also the actual values.

9.

With $h = 0.1$:	$x(1) \approx 3.99261$	$y(1) \approx 6.21770$
With $h = 0.05$:	$x(1) \approx 3.99234$	$y(1) \approx 6.21768$
Actual values:	$x(1) \approx 3.99232$	$y(1) \approx 6.21768$

10.

With $h = 0.1$:	$x(1) \approx 1.31498$	$y(1) \approx 1.02537$
With $h = 0.05$:	$x(1) \approx 1.31501$	$y(1) \approx 1.02538$
Actual values:	$x(1) \approx 1.31501$	$y(1) \approx 1.02538$

11.

With $h = 0.1$:	$x(1) \approx -0.05832$	$y(1) \approx 0.56664$
With $h = 0.05$:	$x(1) \approx -0.05832$	$y(1) \approx 0.56665$
Actual values:	$x(1) \approx -0.05832$	$y(1) \approx 0.56665$

12. We first convert the given initial value problem to the two-dimensional problem

$$x' = y, \qquad x(0) = 0,$$
$$y' = -x + \sin t, \quad y(0) = 0.$$

Then with both step sizes $h = 0.1$ and $h = 0.05$ we get the actual value $x(1) \approx 0.15058$ accurate to 5 decimal places.

13. With $y = x'$ we want to solve numerically the initial value problem

$$x' = y, \qquad x(0) = 0,$$
$$y' = -32 - 0.04y, \quad y(0) = 288.$$

When we run Program RK2DIM with step size $h = 0.1$ we find that the change of sign in the velocity v occurs as follows:

t	x	v
7.6	1050.2	+2.8
7.7	1050.3	-0.4

Thus the bolt attains a maximum height of about 1050 feet in about 7.7 seconds.

14. Now we want to solve numerically the initial value problem

$$x' = y, \qquad x(0) = 0,$$
$$y' = -32 - 0.0002y^2, \quad y(0) = 288.$$

Running Program RK2DIM with step size $h = 0.1$, we find that the bolt attains a maximum height of about 1044 ft in about 7.8 sec. Note that these values are comparable to those found in Problem 13.

15. With $y = x'$ and with x in miles and in seconds, we want to solve numerically the initial value problem

$$x' = y, \qquad x(0) = 0,$$
$$y' = \frac{-95485.5}{x^2 + 7920x + 15681600}, \qquad y(0) = 1.$$

We find (running RK2DIM with $h = 1$) that the projectile reaches a maximum height of about 83.83 miles in about $168 \text{ sec} = 2 \text{ min } 48 \text{ sec}$.

16. We first defined the MATLAB function

```
function  xp  =  fnball(t,x)
%  Defines the baseball system
%       x1′  =  x′  =  x3,  x3′  =  -cvx′
%       x2′  =  y′  =  x4,  x4′  =  -cvy′- g
%  with air resistance coefficient c.

g   =  32;
c   =  0.0025;
xp  =  x;
v  =  sqrt(x(3).^2) + x(4).^2);
xp(1)   =   x(3);
xp(2)   =   x(4);
xp(3)   =  -c*v*x(3);
xp(4)   =  -c*v*x(4) - g;
```

Then, using the n-dimensional program **rkn** with step size 0.1 and initial data corresponding to the indicated initial inclination angles, we got the following results:

Angle	Time	Range
40	5.0	352.9
45	5.4	347.2
50	5.8	334.2

We have listed the time to the nearest tenth of a second, but have interpolated to find the range in feet.

17. The data in Problem 16 indicate that the range increases when the initial angle is decreased below $45°$. The further data

Angle	Range
41.0	352.1
40.5	352.6
40.0	352.9
39.5	352.8
39.0	352.7
35.0	350.8

indicate that a maximum range of about 353 ft is attained with $\alpha \approx 40°$.

18. We "shoot" for the proper inclination angle by running program **rkn** (with $h = 0.1$) as follows:

Angle	Range
60	287.1
58	298.5
57.5	301.1

Thus we get a range of 300 ft with an initial angle just under $57.5°$.

19. First we run program **rkn** (with $h = 0.1$) with $v_0 = 250$ ft/sec and obtain the following results:

t	x	y
5.0	457.43	103.90
6.0	503.73	36.36

Interpolation gives $x = 494.4$ when $y = 50$. Then a run with $v_0 = 255$ ft/sec gives the following results:

t	x	y
5.5	486.75	77.46
6.0	508.86	41.62

Finally, a run with $v_0 = 253$ ft/sec gives these results:

t	x	y
5.5	484.77	75.44
6.0	506.82	39.53

Now $x \approx 500\,\text{ft}$ when $y = 50\,\text{ft}$. Thus Babe Ruth's home run ball had an initial velocity of $253\ \text{ft/sec}$.

20. A run of program **`rkn`** with $h = 0.1$ and with the given data yields the following results:

t	x	y	v	α
5.5	989	539	162	+0.95
5.6	1005	539	161	−0.18
⋮	⋮	⋮	⋮	⋮
11.5	1868	16	214	−52
11.6	1881	−1	216	−53

The first two lines of data above indicate that the crossbow bolt attains a maximum height of about 1005 ft in about 5.6 sec. About 6 sec later (total time 11.6 sec) it hits the ground, having traveled about 1880 ft horizontally.

21. A run with $h = 0.1$ indicates that the projectile has a range of about $21,400\,\text{ft} \approx 4.05\,\text{mi}$ and a flight time of about 46 sec. It attains a maximum height of about 8970 ft in about 17.5 sec. At time $t \approx 23\,\text{sec}$ it has its minimum velocity of about 368 ft/sec. It hits the ground ($t \approx 23\,\text{sec}$) at an angle of about $77°$ with a velocity of about 518 ft/sec.