CHAPTER 5

LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

Along with Chapter 4, this chapter is designed to offer considerable flexibility in the treatment of linear systems, depending on the background in linear algebra that students are assumed to have. Sections 4.1 and 4.2 of the previous chapter can stand alone as a brief introduction to linear systems without the use of linear algebra and matrices. But this chapter employs the notation and terminology of elementary linear algebra. For ready reference and review, Section 5.1 includes a complete and self-contained account of the needed background of determinants, matrices, and vectors. The additional linear algebra that is needed in subsequent sections is introduced along the way.

SECTION 5.1

MATRICES AND LINEAR SYSTEMS

The first half-dozen pages of this section are devoted to a review of matrix notation and terminology. With students who've had some prior exposure to matrices and determinants, this review material can be skimmed rapidly. In this event serious study of the section can begin with the subsections on matrix-valued functions and first-order linear systems. About all that's actually needed for this purpose is some acquaintance with determinants, with matrix multiplication and inverse matrices, and with the fact that a square matrix is invertible if and only if its determinant is nonzero.

1. **(a)** $2\mathbf{A}+3\mathbf{B}=\begin{bmatrix}4 & -6\\ 8 & 14\end{bmatrix}+\begin{bmatrix}9 & -12\\ 15 & 3\end{bmatrix}=\begin{bmatrix}13 & -18\\ 23 & 17\end{bmatrix}$

(b) $3\mathbf{A}-2\mathbf{B}=\begin{bmatrix}6 & -9\\ 12 & 21\end{bmatrix}-\begin{bmatrix}6 & -8\\ 10 & 2\end{bmatrix}=\begin{bmatrix}0 & -1\\ 2 & 19\end{bmatrix}$

(c) $\mathbf{AB}=\begin{bmatrix}-9 & -11\\ 47 & -9\end{bmatrix}$ (d) $\mathbf{BA}=\begin{bmatrix}-10 & -37\\ 14 & -8\end{bmatrix}$

2. $(\mathbf{AB})\mathbf{C}=\begin{bmatrix}-9 & -11\\ 47 & -9\end{bmatrix}\begin{bmatrix}0 & 2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}-33 & -7\\ -27 & 103\end{bmatrix}=\begin{bmatrix}2 & -3\\ 4 & 7\end{bmatrix}\begin{bmatrix}-12 & 10\\ 3 & 9\end{bmatrix}=\mathbf{A}(\mathbf{BC})$

$\mathbf{A}(\mathbf{B}+\mathbf{C})=\begin{bmatrix}2 & -3\\ 4 & 7\end{bmatrix}\begin{bmatrix}3 & -2\\ 8 & 0\end{bmatrix}=\begin{bmatrix}-18 & -4\\ 68 & -8\end{bmatrix}=\begin{bmatrix}-9 & -11\\ 47 & -9\end{bmatrix}+\begin{bmatrix}-9 & 7\\ 21 & 1\end{bmatrix}=\mathbf{AB}+\mathbf{AC}$

3. $\mathbf{AB}=\begin{bmatrix}-1 & 8\\ 46 & -1\end{bmatrix}$; $\mathbf{BA}=\begin{bmatrix}11 & -12 & 14\\ -14 & 0 & 7\\ 0 & 8 & -13\end{bmatrix}$

4. $\mathbf{Ay}=\begin{bmatrix}2t^2-\cos t\\ 3t^2-4\sin t+5\cos t\end{bmatrix}$; $\mathbf{Bx}=\begin{bmatrix}2t+3e^{-t}\\ -14t\\ 6t-2e^{-t}\end{bmatrix}$

The products $\mathbf{Ax}$ and $\mathbf{By}$ are not defined, because in neither case is the number of columns of the first factor equal to the number of rows of the second factor.

5. **(a)** $7\mathbf{A}+4\mathbf{B}=\begin{bmatrix}21 & 14 & -7\\ 0 & 28 & 21\\ -35 & 14 & 49\end{bmatrix}+\begin{bmatrix}0 & -12 & 8\\ 4 & 16 & -12\\ 8 & 20 & -4\end{bmatrix}=\begin{bmatrix}21 & 2 & 1\\ 4 & 44 & 9\\ -27 & 34 & 45\end{bmatrix}$

(b) $3\mathbf{A}-5\mathbf{B}=\begin{bmatrix}9 & 6 & -3\\ 0 & 12 & 9\\ -15 & 6 & 21\end{bmatrix}-\begin{bmatrix}0 & -15 & 10\\ 5 & 20 & -15\\ 10 & 25 & -5\end{bmatrix}=\begin{bmatrix}9 & 21 & -13\\ -5 & -8 & 24\\ -25 & -19 & 26\end{bmatrix}$

(c) $\mathbf{AB}=\begin{bmatrix}0 & -6 & 1\\ 10 & 31 & -15\\ 16 & 58 & -23\end{bmatrix}$ (d) $\mathbf{BA}=\begin{bmatrix}-10 & -8 & 5\\ 18 & 12 & -10\\ 11 & 22 & 6\end{bmatrix}$

(e) $\mathbf{A}-t\mathbf{I}=\begin{bmatrix}3 & 2 & -1\\ 0 & 4 & 3\\ -5 & 2 & 7\end{bmatrix}-\begin{bmatrix}t & 0 & 0\\ 0 & t & 0\\ 0 & 0 & t\end{bmatrix}=\begin{bmatrix}3-t & 2 & -1\\ 0 & 4-t & 3\\ -5 & 2 & 7-t\end{bmatrix}$

6. **(a)** $\mathbf{A}_1\mathbf{B}=\mathbf{A}_2\mathbf{B}=\begin{bmatrix}5 & 10\\ -4 & -8\end{bmatrix}$ **(b)** $\mathbf{AB}=\begin{bmatrix}1 & -2\\ -2 & 4\end{bmatrix}\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{0}$

7. $\det(\mathbf{AB})=0=7\cdot 0=\det(\mathbf{A})\cdot\det(\mathbf{B})$

8. $\det(\mathbf{AB})=\det(\mathbf{BA})=144$ (with $\mathbf{AB}$ and $\mathbf{BA}$ as in Problem 5)

9. $(\mathbf{AB})'=\begin{bmatrix}t-4t^2+6t^3 & t+t^2-4t^3+8t^4\\ 3t+t^3-t^4 & 4t^2+t^3+t^4\end{bmatrix}'=\begin{bmatrix}1-8t+18t^2 & 1+2t-12t^2+32t^3\\ 3+3t^2-4t^3 & 8t+3t^2+4t^3\end{bmatrix}$

$$\mathbf{A}'\mathbf{B}+\mathbf{A}\mathbf{B}'=\begin{bmatrix}1 & 2\\ 3t^2 & -\dfrac{1}{t^2}\end{bmatrix}\begin{bmatrix}1-t & 1+t\\ 3t^2 & 4t^3\end{bmatrix}+\begin{bmatrix}t & 2t-1\\ t^3 & \dfrac{1}{t}\end{bmatrix}\begin{bmatrix}-1 & 1\\ 6t & 12t^2\end{bmatrix}$$

$$=\begin{bmatrix}1-t+6t^2 & 1+t+8t^3\\ -3+3t^2-3t^3 & -4t+3t^2+3t^3\end{bmatrix}+\begin{bmatrix}-7t+12t^2 & t-12t^2+24t^3\\ 6-t^3 & 12t+t^3\end{bmatrix}$$

$$=\begin{bmatrix}1-8t+18t^2 & 1+2t-12t^2+32t^3\\ 3+3t^2-4t^3 & 8t+3t^2+4t^3\end{bmatrix}$$

10. $$(\mathbf{AB})'=\begin{bmatrix}3t^3+3e^t+2te^{-t}\\ 3t\\ 3t^4+24t-2e^{-t}\end{bmatrix}'=\begin{bmatrix}9t^2+3e^t+2e^{-t}-2te^{-t}\\ 3\\ 12t^3+24+2e^{-t}\end{bmatrix}$$

$$\mathbf{A}'\mathbf{B}+\mathbf{A}\mathbf{B}'=\begin{bmatrix}e^t & 1 & 2t\\ -1 & 0 & 0\\ 8 & 0 & 3t^2\end{bmatrix}\begin{bmatrix}3\\ 2e^{-t}\\ 3t\end{bmatrix}+\begin{bmatrix}e^t & t & t^2\\ -t & 0 & 2\\ 8t & -1 & t^3\end{bmatrix}\begin{bmatrix}0\\ -2e^{-t}\\ 3\end{bmatrix}$$

$$=\begin{bmatrix}6t^2+3e^t+2e^{-t}\\ -3\\ 24+9t^3\end{bmatrix}+\begin{bmatrix}3t^2-2te^{-t}\\ 6\\ 3t^3+2e^{-t}\end{bmatrix}$$

$$=\begin{bmatrix}9t^2+3e^t+2e^{-t}-2te^{-t}\\ 3\\ 12t^3+24+2e^{-t}\end{bmatrix}$$

11. $\mathbf{x}=\begin{bmatrix}x\\ y\end{bmatrix}$; $\mathbf{P}(t)=\begin{bmatrix}0 & -3\\ 3 & 0\end{bmatrix}$; $\mathbf{f}(t)=\begin{bmatrix}0\\ 0\end{bmatrix}$

12. $\mathbf{x}=\begin{bmatrix}x\\ y\end{bmatrix}$; $\mathbf{P}(t)=\begin{bmatrix}3 & -2\\ 2 & 1\end{bmatrix}$; $\mathbf{f}(t)=\begin{bmatrix}0\\ 0\end{bmatrix}$

13. $\mathbf{x}=\begin{bmatrix}x\\ y\end{bmatrix}$; $\mathbf{P}(t)=\begin{bmatrix}2 & 4\\ 5 & -1\end{bmatrix}$; $\mathbf{f}(t)=\begin{bmatrix}3e^t\\ -t^2\end{bmatrix}$

14. $\mathbf{x}=\begin{bmatrix}x\\ y\end{bmatrix}$; $\mathbf{P}(t)=\begin{bmatrix}t & -e^t\\ e^{-t} & t^2\end{bmatrix}$; $\mathbf{f}(t)=\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}$

15. $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix};\ \mathbf{P}(t)=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix};\ \mathbf{f}(t)=\begin{bmatrix}0\\0\\0\end{bmatrix}$

16. $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix};\ \mathbf{P}(t)=\begin{bmatrix}2&-3&0\\1&1&2\\0&5&-7\end{bmatrix};\ \mathbf{f}(t)=\begin{bmatrix}0\\0\\0\end{bmatrix}$

17. $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix};\ \mathbf{P}(t)=\begin{bmatrix}3&-4&1\\1&0&-3\\0&6&-7\end{bmatrix};\ \mathbf{f}(t)=\begin{bmatrix}t\\t^2\\t^3\end{bmatrix}$

18. $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix};\ \mathbf{P}(t)=\begin{bmatrix}t&-1&e^t\\2&t^2&-1\\e^{-t}&3t&t^3\end{bmatrix};\ \mathbf{f}(t)=\begin{bmatrix}0\\0\\0\end{bmatrix}$

19. $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix};\ \mathbf{P}(t)=\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\4&0&0&0\end{bmatrix};\ \mathbf{f}(t)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

20. $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix};\ \mathbf{P}(t)=\begin{bmatrix}0&1&1&0\\0&0&1&1\\1&0&0&1\\1&1&0&0\end{bmatrix};\ \mathbf{f}(t)=\begin{bmatrix}0\\t\\t^2\\t^3\end{bmatrix}$

21. $W(t)=\begin{vmatrix}2e^t&e^{2t}\\-3e^t&-e^{2t}\end{vmatrix}=e^{3t}\neq 0;$

$$\mathbf{x}_1'=\begin{bmatrix}2e^t\\-3e^t\end{bmatrix}'=\begin{bmatrix}2e^t\\-3e^t\end{bmatrix}=\begin{bmatrix}4&2\\-3&-1\end{bmatrix}\begin{bmatrix}2e^t\\-3e^t\end{bmatrix}=\mathbf{A}\mathbf{x}_1$$

$$\mathbf{x}_2'=\begin{bmatrix}e^{2t}\\-e^{2t}\end{bmatrix}'=\begin{bmatrix}2e^{2t}\\-2e^{2t}\end{bmatrix}=\begin{bmatrix}4&2\\-3&-1\end{bmatrix}\begin{bmatrix}e^{2t}\\-e^{2t}\end{bmatrix}=\mathbf{A}\mathbf{x}_2$$

$$\mathbf{x}(t)=c_1\mathbf{x}_1+c_2\mathbf{x}_2=c_1\begin{bmatrix}2e^t\\-3e^t\end{bmatrix}+c_2\begin{bmatrix}e^{2t}\\-e^{2t}\end{bmatrix}=\begin{bmatrix}2c_1e^t+c_2e^{2t}\\-3c_1e^t-c_2e^{2t}\end{bmatrix}$$

In most of Problems 22–30, we omit the verifications of the given solutions. In each case, this is simply a matter of calculating both the derivative $\mathbf{x}_i'$ of the given solution vector and the product $\mathbf{Ax}_i$ (where $\mathbf{A}$ is the coefficient matrix in the given differential equation) to verify that $\mathbf{x}_i' = \mathbf{Ax}_i$ (just as in the verification of the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ in Problem 21 above).

22. $$W(t) = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = -5e^{t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + c_2\begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} c_1e^{3t} + 2c_2e^{-2t} \\ 3c_1e^{3t} + c_2e^{-2t} \end{bmatrix}$$

23. $$W(t) = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 5 \end{bmatrix}e^{-2t} = \begin{bmatrix} c_1e^{2t} + c_2e^{-2t} \\ c_1e^{2t} + 5c_2e^{-2t} \end{bmatrix}$$

24. $$W(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{vmatrix} = -e^{5t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{2t} = \begin{bmatrix} c_1e^{3t} + c_2e^{2t} \\ -c_1e^{3t} - 2c_2e^{2t} \end{bmatrix}$$

25. $$W(t) = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 7e^{-3t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} 3c_1e^{2t} + c_2e^{-5t} \\ 2c_1e^{2t} + 3c_2e^{-5t} \end{bmatrix}$$

26. $$W(t) = \begin{vmatrix} 2e^{t} & -2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & e^{3t} & e^{5t} \end{vmatrix} = 16e^{9t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}e^{3t} + c_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}e^{5t} = \begin{bmatrix} 2c_1e^{t} - 2c_2e^{3t} + 2c_3e^{5t} \\ 2c_1e^{t} - 2c_3e^{5t} \\ c_1e^{t} + c_2e^{3t} + c_3e^{5t} \end{bmatrix}$$

27. $$W(t)=\begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix}=3\neq 0$$

$$\mathbf{x}(t)=c_1\mathbf{x}_1+c_2\mathbf{x}_2+c_3\mathbf{x}_3=c_1\begin{bmatrix}1\\1\\1\end{bmatrix}e^{2t}+c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-t}+c_3\begin{bmatrix}0\\1\\-1\end{bmatrix}e^{-t}=\begin{bmatrix}c_1e^{2t}+c_2e^{-t}\\c_1e^{2t}+c_3e^{-t}\\c_1e^{2t}-c_2e^{-t}-c_3e^{-t}\end{bmatrix}$$

$$\mathbf{x}_1'=\begin{bmatrix}2\\2\\2\end{bmatrix}e^{t}=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}e^{t}=\mathbf{Ax}_1;\quad \mathbf{x}_2'=\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t}=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-t}=\mathbf{Ax}_2;$$

$$\mathbf{x}_3'=\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{t}=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix}e^{-t}=\mathbf{Ax}_3$$

28. $$W(t)=\begin{vmatrix} 1 & 2e^{3t} & -e^{4t} \\ 6 & 3e^{3t} & 2e^{4t} \\ -13 & -2e^{3t} & e^{4t} \end{vmatrix}=-84e^{-t}\neq 0$$

$$\mathbf{x}(t)=c_1\mathbf{x}_1+c_2\mathbf{x}_2+c_3\mathbf{x}_3=c_1\begin{bmatrix}1\\6\\-13\end{bmatrix}+c_2\begin{bmatrix}2\\3\\-2\end{bmatrix}e^{3t}+c_3\begin{bmatrix}-1\\2\\1\end{bmatrix}e^{4t}=\begin{bmatrix}c_1+2c_2e^{3t}-c_3e^{4t}\\6c_1+3c_2e^{3t}+2c_3e^{4t}\\-13c_1-2c_2e^{3t}+c_3e^{4t}\end{bmatrix}$$

29. $$W(t)=\begin{vmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{vmatrix}=e^{2t}\neq 0$$

$$\mathbf{x}(t)=c_1\mathbf{x}_1+c_2\mathbf{x}_2+c_3\mathbf{x}_3=c_1\begin{bmatrix}3\\-2\\2\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{t}+c_3\begin{bmatrix}1\\-1\\0\end{bmatrix}e^{3t}=\begin{bmatrix}3c_1e^{-2t}+c_2e^{t}+c_3e^{3t}\\-2c_1e^{-2t}-c_2e^{t}-c_3e^{3t}\\2c_1e^{-2t}+c_2e^{t}\end{bmatrix}$$

30. $$W(t)=\begin{vmatrix} e^{-t} & 0 & 0 & e^{t} \\ 0 & 0 & e^{t} & 0 \\ 0 & e^{-t} & 0 & 3e^{t} \\ e^{-t} & 0 & -2e^{t} & 0 \end{vmatrix}=-e^{-t}\begin{vmatrix} e^{-t} & 0 & e^{t} \\ 0 & e^{t} & 0 \\ e^{-t} & -2e^{t} & 0 \end{vmatrix}=-\begin{vmatrix} 0 & e^{t} \\ e^{1t} & -2e^{t} \end{vmatrix}=1\neq 0$$

$$\mathbf{x}(t)=c_1\begin{bmatrix}1\\0\\0\\1\end{bmatrix}e^{-t}+c_2\begin{bmatrix}0\\0\\1\\0\end{bmatrix}e^{-t}+c_3\begin{bmatrix}0\\1\\0\\-2\end{bmatrix}e^{t}+c_4\begin{bmatrix}1\\0\\3\\0\end{bmatrix}e^{t}=\begin{bmatrix}c_1e^{-t}+c_4e^t\\c_3e^t\\c_2e^{-t}+3c_4e^t\\c_1e^{-t}-2c_3e^t\end{bmatrix}$$

In Problems 31–34 (and similarly in Problems 35–40) we give first the scalar components $x_1(t)$ and $x_2(t)$ of a general solution, then the equations in the coefficients c_1 and c_2 that are obtained when the given initial conditions are imposed, and finally the resulting particular solution of the given system.

31. $x_1(t)=c_1e^{3t}+2c_2e^{-2t},\quad x_2(t)=3c_1e^{3t}+c_2e^{-2t}$

$$c_1+2c_2=0,\qquad 3c_1+c_2=5$$
$$x_1(t)=2e^{3t}-2e^{-2t},\quad x_2(t)=6e^{3t}-e^{-2t}$$

32. $x_1(t)=c_1e^{2t}+c_2e^{-2t},\quad x_2(t)=c_1e^{2t}+5c_2e^{-2t}$

$$c_1+c_2=5,\qquad c_1+5c_2=-3$$
$$x_1(t)=7e^{2t}-2e^{-2t},\quad x_2(t)=7e^{2t}-10e^{-2t}$$

33. $x_1(t)=c_1e^{3t}+c_2e^{2t},\quad x_2(t)=-c_1e^{3t}-2c_2e^{2t}$

$$c_1+c_2=11,\qquad -c_1-2c_2=-7$$
$$x_1(t)=15e^{3t}-4e^{2t},\quad x_2(t)=-15e^{3t}+8e^{2t}$$

34. $x_1(t)=3c_1e^{2t}+c_2e^{-5t},\quad x_2(t)=2c_1e^{2t}+3c_2e^{-5t}$

$$3c_1+c_2=8,\qquad 2c_1+3c_2=0$$
$$x_1(t)=\frac{8}{7}\left(9e^{2t}-2e^{-5t}\right),\quad x_2(t)=\frac{48}{7}\left(e^{2t}-e^{-5t}\right)$$

35. $x_1(t)=2c_1e^{t}-2c_2e^{3t}+2c_3e^{5t},\quad x_2(t)=2c_1e^{t}-2c_3e^{5t},\quad x_3(t)=c_1e^{t}+c_2e^{3t}+c_3e^{5t}$

$$2c_1-2c_2+2c_3=0,\qquad 2c_1-2c_3=0,\qquad c_1+c_2+c_3=4$$
$$x_1(t)=2e^{t}-4e^{3t}+2e^{5t},\quad x_2(t)=2e^{t}-2e^{5t},\quad x_3(t)=e^{t}+2e^{3t}+e^{5t}$$

36. $x_1(t)=c_1e^{2t}+c_2e^{-t},\quad x_2(t)=c_1e^{2t}+c_3e^{-t},\quad x_3(t)=c_1e^{2t}-c_2e^{-t}-c_3e^{-t}$

$$c_1+c_2=10,\qquad c_1+c_3=12,\qquad c_1-c_2-c_3=-1$$
$$x_1(t)=7e^{2t}+3e^{-t},\quad x_2(t)=7e^{2t}+5e^{-t},\quad x_3(t)=7e^{2t}-8e^{-t}$$

37. $x_1(t) = 3c_1e^{-2t} + c_2e^t + c_3e^{3t}, \quad x_2(t) = -2c_1e^{-2t} - c_2e^t - c_3e^{3t}, \quad x_3(t) = 2c_1e^{-2t} + c_2e^t$

$$3c_1 + c_2 + c_3 = 1, \qquad -2c_1 - c_2 - c_3 = 2, \qquad 2c_1 + c_2 = 3$$
$$x_1(t) = 9e^{-2t} - 3e^t - 5e^{3t}, \quad x_2(t) = -6e^{-2t} + 3e^t + 5e^{3t}, \quad x_3(t) = 6e^{-2t} - 3e^t$$

38. $x_1(t) = 3c_1e^{-2t} + c_2e^t + c_3e^{3t}, \quad x_2(t) = -2c_1e^{-2t} - c_2e^t - c_3e^{3t}, \quad x_3(t) = 2c_1e^{-2t} + c_2e^t$

$$3c_1 + c_2 + c_3 = 5, \qquad -2c_1 - c_2 - c_3 = -7, \qquad 2c_1 + c_2 = 11$$
$$x_1(t) = -6e^{-2t} + 15e^t - 4e^{3t}, \quad x_2(t) = 4e^{-2t} - 15e^t + 4e^{3t}, \quad x_3(t) = -4e^{-2t} + 15e^t$$

39. $x_1(t) = c_1e^{-t} + c_4e^t, \quad x_2(t) = c_3e^t, \quad x_3(t) = c_2e^{-t} + 3c_4e^t, \quad x_4(t) = c_1e^{-t} - 2c_3e^t$

$$c_1 + c_4 = 1, \qquad c_3 = 1, \qquad c_2 + 3c_4 = 1, \qquad c_1 - 2c_3 = 1$$
$$x_1(t) = 3e^{-t} - 2e^t, \quad x_2(t) = e^t, \quad x_3(t) = 7e^{-t} - 6e^t, \quad x_4(t) = 3e^{-t} - 2e^t$$

40. $x_1(t) = c_1e^{-t} + c_4e^t, \quad x_2(t) = c_3e^t, \quad x_3(t) = c_2e^{-t} + 3c_4e^t, \quad x_4(t) = c_1e^{-t} - 2c_3e^t$

$$c_1 + c_4 = 1, \qquad c_3 = 3, \qquad c_2 + 3c_4 = 4, \qquad c_1 - 2c_3 = 7$$
$$x_1(t) = 13e^{-t} - 12e^t, \quad x_2(t) = 3e^t, \quad x_3(t) = 40e^{-t} - 36e^t, \quad x_4(t) = 13e^{-t} - 6e^t$$

41. **(a)** $\mathbf{x}_2 = t\mathbf{x}_1$, so neither is a constant multiple of the other.

(b) $W(\mathbf{x}_1, \mathbf{x}_2) = 0$, whereas Theorem 2 would imply that $W \neq 0$ if $\mathbf{x}_1$ and $\mathbf{x}_2$ were independent solutions of a system of the indicated form.

42. If $x_{12}(t) = cx_{11}(t)$ and $x_{22}(t) = cx_{21}(t)$, then

$$W(t) = x_{11}(t)x_{22}(t) - x_{12}(t)x_{21}(t) = cx_{11}(t)x_{21}(t) - cx_{11}(t)x_{21}(t) = 0.$$

43. Suppose $W(a) = x_{11}(a)x_{22}(a) - x_{12}(a)x_{21}(a) = 0$. Then the coefficient determinant of the homogeneous linear system

$$c_1x_{11}(a) + c_2x_{12}(a) = 0, \quad c_1x_{21}(a) + c_2x_{22}(a) = 0$$

vanishes. The system therefore has a non-trivial solution $\{c_1, c_2\}$ such that $\mathbf{x}(a) = \mathbf{0}$. It therefore follows (by uniqueness of solutions) that $\mathbf{x}(t) \equiv \mathbf{0}$, that is, $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \equiv \mathbf{0}$ with c_1 and c_2 not both zero. Thus the solution vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent.

44. The argument is precisely the same, except with n solution vectors each having n component functions (rather than 2 solution vectors each having 2 component functions).

45. Suppose that $c_1\mathbf{x}_1(t)+c_2\mathbf{x}_2(t)+\cdots+c_n\mathbf{x}_n(t)\equiv\mathbf{0}$. Then the i^{th} scalar component of this vector equation is $c_1x_{i1}(t)+c_2x_{i2}(t)+\cdots+c_nx_{in}(t)\equiv 0$. Hence the fact that the scalar functions $x_{i1}(t)$, $x_{i2}(t)$, $x_{in}(t)$ are linearly independent implies that $c_1=c_2=\cdots=c_n=0$. Consequently the vector functions $\mathbf{x}_1(t),\mathbf{x}_2(t),\cdots,\mathbf{x}_n(t)$ are linearly independent.

SECTION 5.2

THE EIGENVALUE METHOD FOR HOMOGENEOUS LINEAR SYSTEMS

In each of Problems 1–16 we give the characteristic equation, the eigenvalues λ_1 and λ_2 of the coefficient matrix of the given system, the corresponding equations determining the associated eigenvectors $\mathbf{v}_1=\begin{bmatrix}a_1 & b_1\end{bmatrix}^{\mathrm{T}}$ and $\mathbf{v}_2=\begin{bmatrix}a_2 & b_2\end{bmatrix}^{\mathrm{T}}$, these eigenvectors, and the resulting scalar components $x_1(t)$ and $x_2(t)$ of a general solution $\mathbf{x}(t)=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}$ of the system. Finally, the figure for each Problem shows a direction field and some typical solution curves for the system.

1. Characteristic equation $\lambda^2-2\lambda-3=0$;

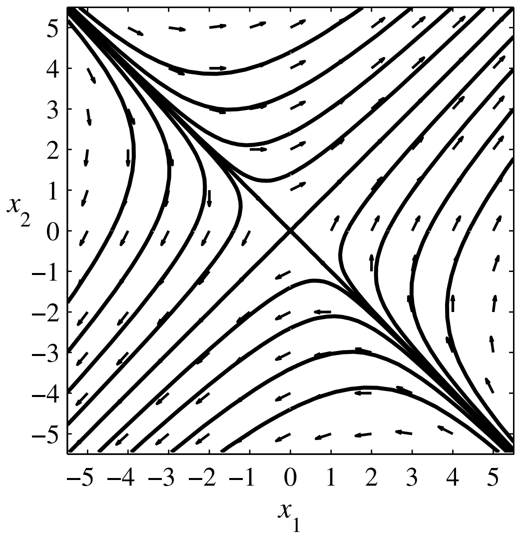
Eigenvalues $\lambda_1=-1$ and $\lambda_2=3$;

Eigenvector equations $\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}a_1\\ b_1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ and $\begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix}\begin{bmatrix}a_2\\ b_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$;

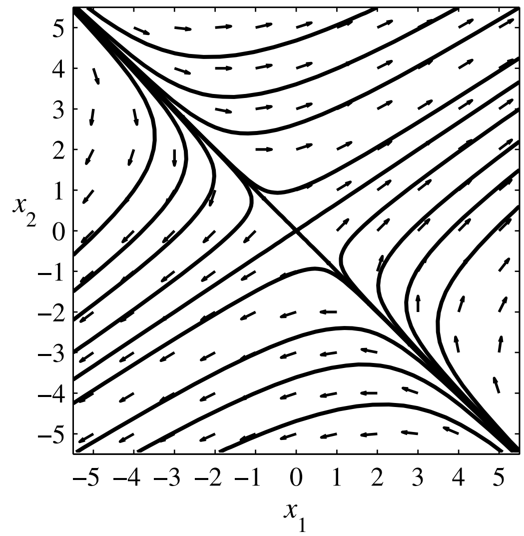
Eigenvectors $\mathbf{v}_1=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$ and $\mathbf{v}_2=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$;

$$x_1(t)=c_1e^{-t}+c_2e^{3t},\quad x_2(t)=-c_1e^{-t}+c_2e^{3t}$$

Problem 1



Problem 2



2. Characteristic equation $\lambda^2 - 3\lambda - 4 = 0$;

Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$;

Eigenvector equations $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$;

$$x_1(t) = c_1 e^{-t} + 3c_2 e^{4t}, \quad x_2(t) = -c_1 e^{-t} + 2c_2 e^{4t}$$

3. Characteristic equation $\lambda^2 - 5\lambda - 6 = 0$;

Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 6$;

Eigenvector equations $\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $\mathbf{v}_1 = \begin{bmatrix} 4 & 3 \end{bmatrix}^T$;

$$x_1(t) = c_1 e^{-t} + 4c_2 e^{6t}, \quad x_2(t) = -c_1 e^{-t} + 3c_2 e^{6t};$$

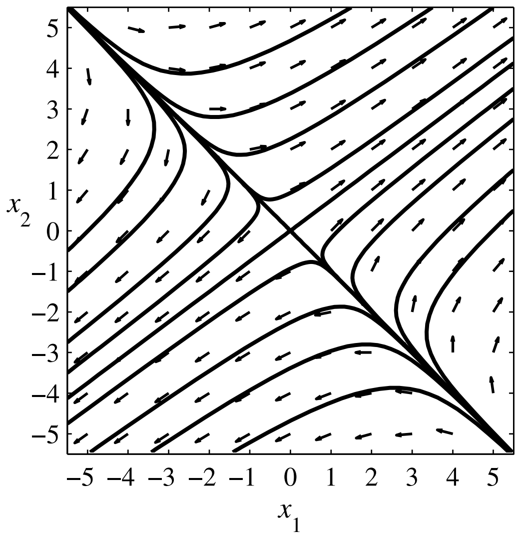
The equations

$$x_1(0) = c_1 + 4c_2 = 1, \quad x_2(0) = -c_1 + 3c_2 = 1$$

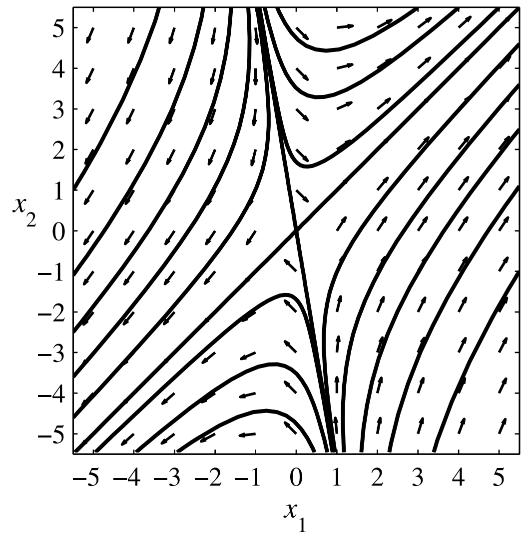
yield $c_1 = -\frac{1}{7}$ and $c_2 = \frac{2}{7}$, so the desired particular solution is given by

$$x_1(t) = \frac{1}{7}\left(-e^{-t} + 8e^{6t}\right), \quad x_2(t) = \frac{1}{7}\left(e^{-t} + 6e^{6t}\right).$$

Problem 3



Problem 4

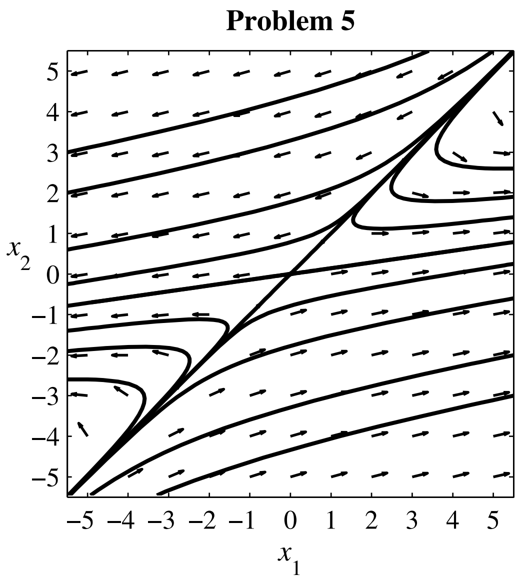


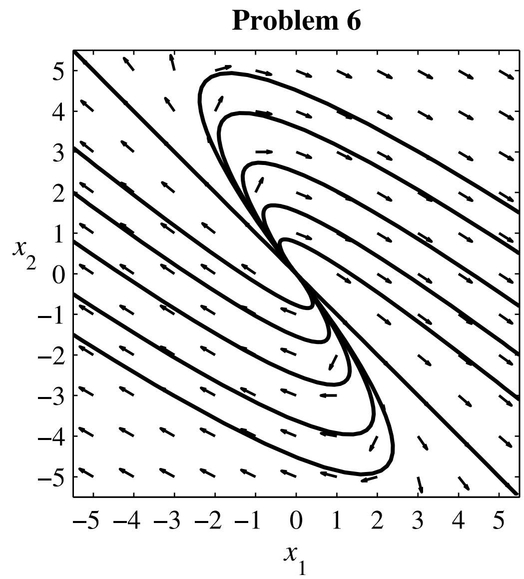
4. Characteristic equation $\lambda^2 - 3\lambda - 10 = 0$;
Eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$;
Eigenvector equations $\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & -6 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$;

$$x_1(t) = c_1 e^{-2t} + c_2 e^{5t}, \quad x_2(t) = -6c_1 e^{-2t} + c_2 e^{5t}$$

5. Characteristic equation $\lambda^2 - 4\lambda - 5 = 0$;
Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$;
Eigenvector equations $\begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 7 & 1 \end{bmatrix}^T$;

$$x_1(t) = c_1 e^{-t} + 7c_2 e^{5t}, \quad x_2(t) = c_1 e^{-t} + c_2 e^{5t}$$





6. Characteristic equation $\lambda^2 - 7\lambda + 12 = 0$;
Eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 4$;
Eigenvector equations $\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 5 & -6 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$;

$$x_1(t) = 5c_1e^{3t} + c_2e^{4t}, \quad x_2(t) = -6c_1e^{3t} - c_2e^{4t}$$

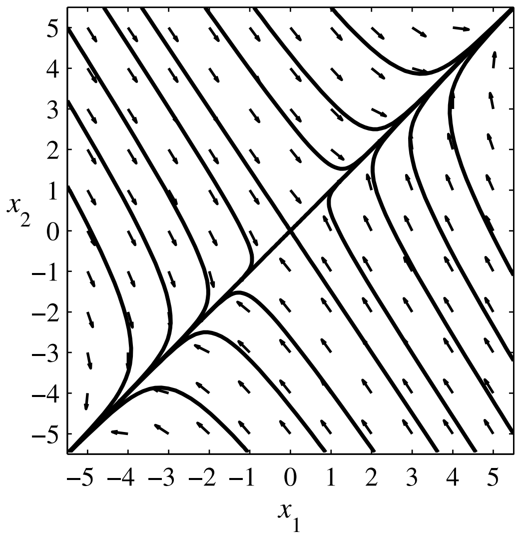
The initial conditions yield $c_1 = -1$ and $c_2 = 6$, so

$$x_1(t) = -5e^{3t} + 6e^{4t}, \quad x_2(t) = 6e^{3t} - 6e^{4t}$$

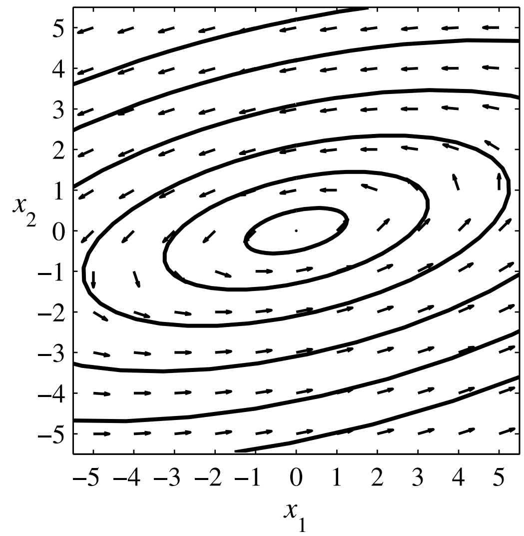
7. Characteristic equation $\lambda^2 + 8\lambda - 9 = 0$;
Eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -9$;
Eigenvector equations $\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 2 & -3 \end{bmatrix}^T$;

$$x_1(t) = c_1e^{t} + 2c_2e^{-9t}, \quad x_2(t) = c_1e^{t} - 3c_2e^{-9t}$$

Problem 7



Problem 8



8. Characteristic equation $\lambda^2 + 4 = 0$;
Eigenvalue $\lambda = 2i$;
Eigenvector equation $\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvector $\mathbf{v} = \begin{bmatrix} 5 & 1-2i \end{bmatrix}^T$;

$$\mathbf{x}(t) = \mathbf{v}e^{2it} = \begin{bmatrix} 5\cos 2t + 5i\sin 2t \\ (\cos 2t + 2\sin 2t) + i(\sin 2t - 2\cos 2t) \end{bmatrix};$$

$$x_1(t) = 5c_1\cos 2t + 5c_2\sin 2t,$$

$$x_2(t) = c_1(\cos 2t + 2\sin 2t) + c_2(\sin 2t - 2\cos 2t) = (c_1 - 2c_2)\cos 2t + (2c_1 + c_2)\sin 2t$$

9. Characteristic equation $\lambda^2 + 16 = 0$;
Eigenvalue $\lambda = 4i$;
Eigenvector equation $\begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvector $\mathbf{v} = \begin{bmatrix} 5 & 2-4i \end{bmatrix}^T$;
The real and imaginary parts of

$$\mathbf{x}(t) = \mathbf{v}e^{4it} = \begin{bmatrix} 5\cos 4t + 5i\sin 4t \\ (2\cos 4t + 4\sin 4t) + i(2\sin 4t - 4\cos 4t) \end{bmatrix}$$

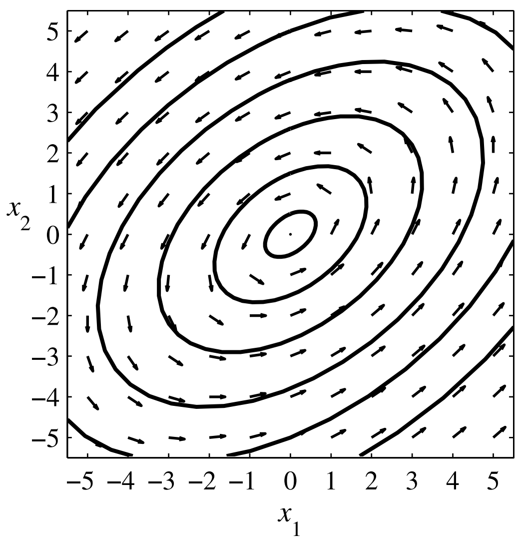
yield the general solution

$$x_1(t) = 5c_1 \cos 4t + 5c_2 \sin 4t,$$
$$x_2(t) = c_1(2\cos 4t + 4\sin 4t) + c_2(2\sin 4t - 4\cos 4t)$$

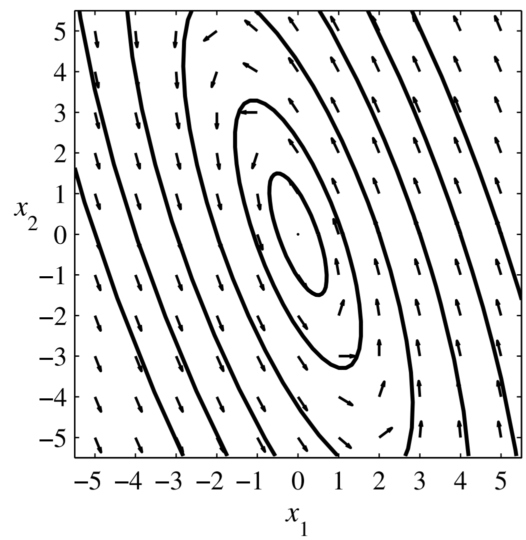
The initial conditions $x_1(0) = 2$ and $x_2(0) = 3$ give $c_1 = \frac{2}{5}$ and $c_2 = -\frac{11}{20}$, so the desired particular solution is

$$x_1(t) = 2\cos 4t - \frac{11}{4}\sin 4t, \quad x_2(t) = 3\cos 4t + \frac{1}{2}\sin 4t$$

Problem 9



Problem 10



10. Characteristic equation $\lambda^2 + 9 = 0$;
Eigenvalue $\lambda = 3i$;
Eigenvector equation $\begin{bmatrix} -3-3i & -2 \\ 9 & 3-3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvector $\mathbf{v} = \begin{bmatrix} -2 & 3+3i \end{bmatrix}^{\mathrm{T}}$;

$$\mathbf{x}(t) = \mathbf{v}e^{3it} = \begin{bmatrix} -2\cos 3t - 2i\sin 3t \\ (3\cos 3t - 3\sin 3t) + i(3\sin 3t + 3\cos 3t) \end{bmatrix};$$

$$x_1(t) = -2c_1 \cos 3t - 2c_2 \sin 3t,$$
$$x_2(t) = c_1(3\cos 3t - 3\sin 3t) + c_2(3\cos 3t + 3\sin 3t)$$
$$= (3c_1 + 3c_2)\cos 3t + (3c_2 - 3c_1)\sin 3t$$

11. Characteristic equation $\lambda^2 - 2\lambda + 5 = 0$;
Eigenvalue $\lambda = 1 - 2i$;

Eigenvector equation $\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

Eigenvector $\mathbf{v} = \begin{bmatrix} 1 & i \end{bmatrix}^{\mathrm{T}}$;

The real and imaginary parts of

$$\mathbf{x}(t) = \begin{bmatrix} 1 & i \end{bmatrix}^{\mathrm{T}} e^t(\cos 2t - i\sin 2t) = e^t\begin{bmatrix} \cos 2t & \sin 2t \end{bmatrix}^{\mathrm{T}} + ie^t\begin{bmatrix} -\sin 2t & \cos 2t \end{bmatrix}^{\mathrm{T}}$$

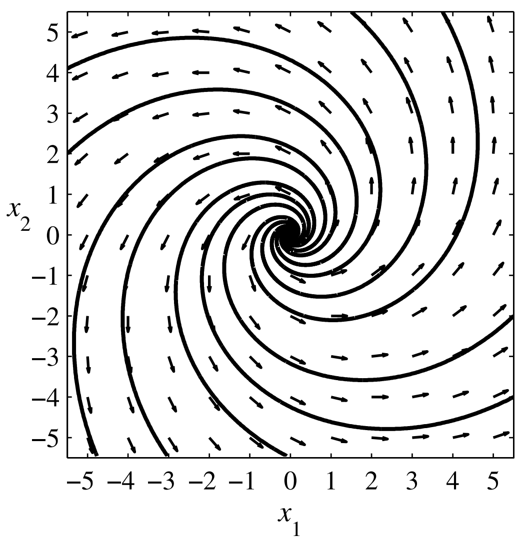
yield the general solution

$$x_1(t) = e^t(c_1\cos 2t - c_2\sin 2t), \quad x_2(t) = e^t(c_1\sin 2t + c_2\cos 2t)$$

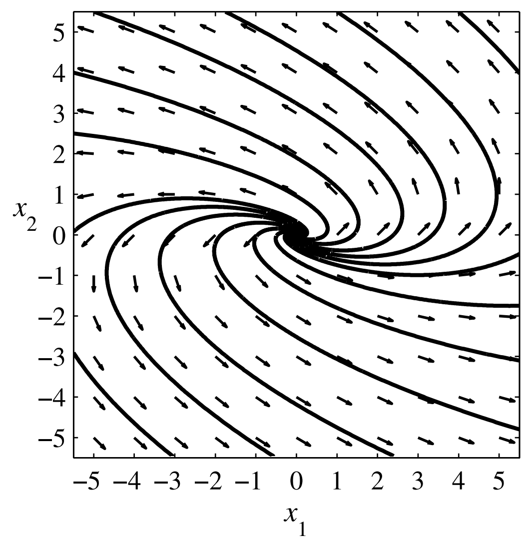
The particular solution with $x_1(0) = 0$ and $x_2(0) = 4$ is obtained with $c_1 = 0$ and $c_2 = 4$, so

$$x_1(t) = -4e^t\sin 2t, \quad x_2(t) = 4e^t\cos 2t\,.$$

Problem 11



Problem 12



12. Characteristic equation $\lambda^2 - 4\lambda + 8 = 0$;

Eigenvalue $\lambda = 2 + 2i$;

Eigenvector equation $\begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

Eigenvector $\mathbf{v} = \begin{bmatrix} -5 & 1+2i \end{bmatrix}^{\mathrm{T}}$;

$$\mathbf{x}(t) = \mathbf{v}e^{(2+2i)t} = e^{2t}\begin{bmatrix} -5\cos 2t - 5i\sin 2t \\ (\cos 2t - 2\sin 2t) + i(\sin 2t + 2\cos 2t) \end{bmatrix};$$

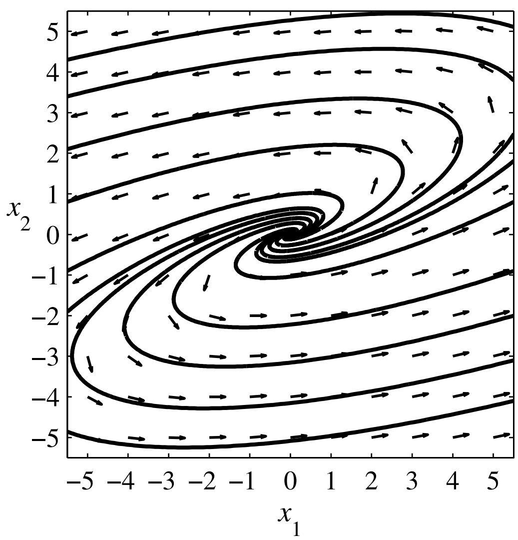
$$x_1(t) = e^{2t}\left(-5c_1\cos 2t - 5c_2\sin 2t\right),$$
$$x_2(t) = e^{2t}\left[c_1\left(\cos 2t - 2\sin 2t\right) + c_2\left(2\cos 2t + \sin 2t\right)\right]$$
$$= e^{2t}\left[\left(c_1 + 2c_2\right)\cos 2t + \left(-2c_1 + c_2\right)\sin 2t\right]$$

13. Characteristic equation $\lambda^2 - 4\lambda + 13 = 0$;
Eigenvalue $\lambda = 2 - 3i$;
Eigenvector equation $\begin{bmatrix} 3+3i & -9 \\ 2 & -3+3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvector $\mathbf{v} = \begin{bmatrix} 3 & 1+i \end{bmatrix}^{\mathrm{T}}$;

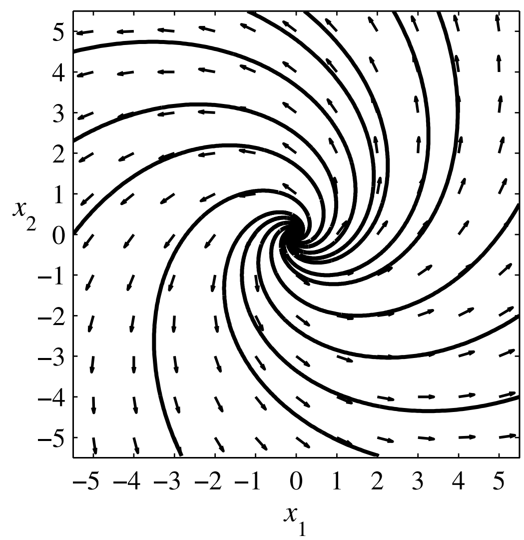
$$\mathbf{x}(t) = \mathbf{v}e^{(2-3i)t} = e^{2t}\begin{bmatrix} 3\cos 3t - 3i\sin 3t \\ \left(\cos 3t + \sin 3t\right) + i\left(\cos 3t - \sin 3t\right) \end{bmatrix}$$

$$x_1(t) = 3e^{2t}\left(c_1\cos 3t - c_2\sin 3t\right), \quad x_2(t) = e^{2t}\left[\left(c_1 + c_2\right)\cos 3t + \left(c_1 - c_2\right)\sin 3t\right]$$

Problem 13



Problem 14



14. Characteristic equation $\lambda^2 - 2\lambda + 5 = 0$;
Eigenvalue $\lambda = 3 + 4i$;
Eigenvector equation $\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvector $\mathbf{v} = \begin{bmatrix} 1 & -i \end{bmatrix}^{\mathrm{T}}$;

$$\mathbf{x}(t) = \mathbf{v}e^{(3+4i)t} = e^{3t}\begin{bmatrix} \cos 4t + i\sin 4t \\ \sin 4t - i\cos 4t \end{bmatrix};$$

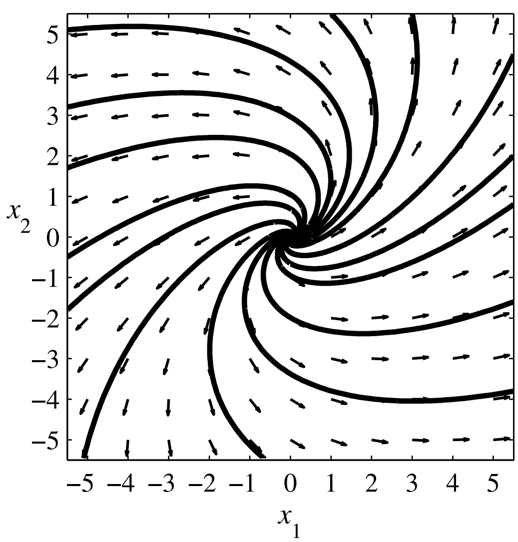
$$x_1(t) = e^{3t}(c_1 \cos 4t + c_2 \sin 4t), \quad x_2(t) = e^{3t}(c_1 \sin 4t - c_2 \cos 4t)$$

15. Characteristic equation $\lambda^2 - 10\lambda + 41 = 0$;
Eigenvalue $\lambda = 5 - 4i$;
Eigenvector equation $\begin{bmatrix} 2+4i & -5 \\ 4 & -2+4i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvector $\mathbf{v} = [5 \quad 2+4i]^{\mathrm{T}}$;

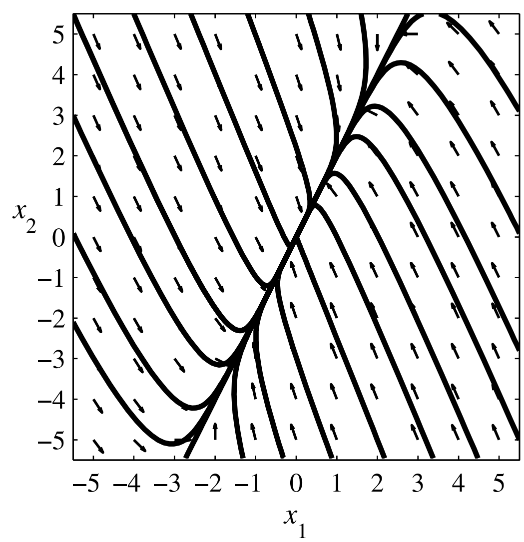
$$\mathbf{x}(t) = \mathbf{v}e^{(5-4i)t} = e^{5t}\begin{bmatrix} 5\cos 4t - 5i \sin 4t \\ (2\cos 4t + 4\sin 4t) + i(4\cos 4t - 2\sin 4t) \end{bmatrix};$$

$$x_1(t) = 5e^{5t}(c_1 \cos 4t - c_2 \sin 4t), \quad x_2(t) = e^{5t}\left[(2c_1 + 4c_2)\cos 4t + (4c_1 - 2c_2)\sin 4t\right]$$

Problem 15



Problem 16



16. Characteristic equation $\lambda^2 + 110\lambda + 1000 = 0$;
Eigenvalues $\lambda_1 = -10$ and $\lambda_2 = -100$;
Eigenvector equations $\begin{bmatrix} -40 & 20 \\ 100 & -50 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 50 & 20 \\ 100 & 40 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
Eigenvectors $\mathbf{v}_1 = [1 \quad 2]^{\mathrm{T}}$ and $\mathbf{v}_2 = [2 \quad -5]^{\mathrm{T}}$;

$$x_1(t) = c_1 e^{-10t} + 2c_2 e^{-100t}, \quad x_2(t) = 2c_1 e^{-10t} - 5c_2 e^{-100t}$$

17. Characteristic equation $-\lambda^3 + 15\lambda^2 - 54\lambda = 0$;
Eigenvalues $\lambda_1 = 9$, $\lambda_2 = 6$, $\lambda_3 = 0$;
Eigenvector equations

$$\begin{bmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_2 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^{\mathrm{T}}$;

$$x_1(t) = c_1 e^{9t} + c_2 e^{6t} + c_3, \quad x_2(t) = c_1 e^{9t} - 2c_2 e^{6t}, \quad x_3(t) = c_1 e^{9t} + c_2 e^{6t} - c_3$$

18. Characteristic equation $-\lambda^3 + 15\lambda^2 - 54\lambda = 0$;

Eigenvalues $\lambda_1 = 9$, $\lambda_2 = 6$, $\lambda_3 = 0$;

Eigenvector equations

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_3 = \begin{bmatrix} 4 & -1 & -1 \end{bmatrix}^{\mathrm{T}}$;

$$x_1(t) = c_1 e^{9t} + 4c_3, \quad x_2(t) = 2c_1 e^{9t} + c_2 e^{6t} - c_3, \quad x_3(t) = 2c_1 e^{9t} - c_2 e^{6t} - c_3$$

19. Characteristic equation $-\lambda^3 + 12\lambda^2 - 45\lambda + 54 = 0$;

Eigenvalues $\lambda_1 = 6$, $\lambda_2 = 3$, $\lambda_3 = 3$;

Eigenvector equations

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_2 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^{\mathrm{T}}$;

$$x_1(t) = c_1 e^{6t} + c_2 e^{3t} + c_3 e^{3t}, \quad x_2(t) = c_1 e^{6t} - 2c_2 e^{3t}, \quad x_3(t) = c_1 e^{6t} + c_2 e^{3t} - c_3 e^{3t}$$

20. Characteristic equation $-\lambda^3 + 17\lambda^2 - 84\lambda + 108 = 0$;

Eigenvalues $\lambda_1 = 9$, $\lambda_2 = 6$, $\lambda_3 = 2$;

Eigenvector equations

$$\begin{bmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_2 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^{\mathrm{T}}$;

$$x_1(t) = c_1e^{9t} + c_2e^{6t} + c_3e^{2t}, \quad x_2(t) = c_1e^{9t} - 2c_2e^{6t}, \quad x_3(t) = c_1e^{9t} + c_2e^{6t} - c_3e^{2t}$$

21. Characteristic equation $-\lambda^3 + \lambda = 0$;
Eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$;
Eigenvector equations

$$\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 6 & 2 & 5 \end{bmatrix}^T$, $\mathbf{v}_2 = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}^T$, $\mathbf{v}_3 = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}^T$;

$$x_1(t) = 6c_1 + 3c_2e^{t} + 2c_3e^{-t}, \quad x_2(t) = 2c_1 + c_2e^{t} + c_3e^{-t}, \quad x_3(t) = 5c_1 + 2c_2e^{t} + 2c_3e^{-t}$$

22. Characteristic equation $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$;
Distinct eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = 3$;
Eigenvector equations

$$\begin{bmatrix} 5 & 2 & 2 \\ -5 & -2 & -2 \\ 5 & 5 & 5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 2 \\ -5 & -5 & -2 \\ 5 & 5 & 2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 2 \\ -5 & -7 & -2 \\ 5 & 5 & 0 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$, $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$, $\mathbf{v}_3 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$;

$$x_1(t) = c_2e^{t} + c_3e^{3t}, \quad x_2(t) = c_1e^{-2t} - c_2e^{t} - c_3e^{3t}, \quad x_3(t) = -c_1e^{-2t} + c_3e^{3t}$$

23. Characteristic equation $-\lambda^3 + 3\lambda^2 + 4\lambda - 12 = 0$;
Eigenvalues $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 3$;
Eigenvector equations

$$\begin{bmatrix} 1 & 1 & 1 \\ -5 & -5 & -1 \\ 5 & 5 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 & 1 & 1 \\ -5 & -1 & -1 \\ 5 & 5 & 5 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ -5 & -6 & -1 \\ 5 & 5 & 0 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$$

Eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$, $\mathbf{v}_3 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$;

$$x_1(t) = c_1e^{2t} + c_3e^{3t}, \quad x_2(t) = -c_1e^{2t} + c_2e^{-2t} - c_3e^{3t}, \quad x_3(t) = -c_2e^{-2t} + c_3e^{3t}$$

24. Characteristic equation $-\lambda^3 + \lambda^2 - 4\lambda + 4 = 0$;
Eigenvalues $\lambda = 1$ and $\lambda = \pm 2i$;

With $\lambda = 1$ the eigenvector equation $\begin{bmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$. To find an eigenvector $\mathbf{v} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ associated with $\lambda = 2i$ we must find a nontrivial solution of the equations

$$\begin{aligned} (2-2i)a &+ & b &- & c &= 0, \\ -4a &+ & (-3-2i)b &- & c &= 0, \\ 4a &+ & 4b &+ & (2-2i)c &= 0. \end{aligned}$$

Subtraction of the first two equations yields

$$(6-2i)a + (4+2i)b = 0,$$

so we take $a = 2+i$ and $b = -3+i$. Then the first equation gives $c = 3-i$. Thus $\mathbf{v} = \begin{bmatrix} 2+i & -3+i & 3-i \end{bmatrix}^T$. Finally

$$\begin{aligned} (2+i)e^{2it} &= (2\cos 2t - \sin 2t) + i(\cos 2t + 2\sin 2t), \\ (3-i)e^{2it} &= (3\cos 2t + \sin 2t) + i(3\sin 2t - \cos 2t), \end{aligned}$$

so the solution is

$$\begin{aligned} x_1(t) &= c_1 e^t + c_2(2\cos 2t - \sin 2t) + c_3(\cos 2t + 2\sin 2t), \\ x_2(t) &= -c_1 e^t - c_2(3\cos 2t + \sin 2t) + c_3(\cos 2t - 3\sin 2t), \\ x_3(t) &= c_2(3\cos 2t + \sin 2t) + c_3(3\sin 2t - \cos 2t). \end{aligned}$$

25. Characteristic equation $-\lambda^3 + 4\lambda^2 - 13\lambda = 0$;
Eigenvalues $\lambda = 0$ and $\lambda = 2 \pm 3i$;
With $\lambda = 1$ the eigenvector equation $\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$. With $\lambda = 2+3i$ we solve the eigenvector equation

$$\begin{bmatrix} 3-3i & 5 & 2 \\ -6 & -8-3i & -5 \\ 6 & 6 & 3-3i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to find the complex-valued eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1+i & -2 & 2 \end{bmatrix}^T$. The corresponding complex-valued solution is

$$\mathbf{x}(t) = \mathbf{v}e^{(2+3i)t} = e^{2t}\begin{bmatrix} (\cos 3t - \sin 3t) + i(\cos 3t + \sin 3t) \\ -2\cos 3t - 2i\sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix}.$$

The scalar components of the resulting general solution are

$$\begin{aligned} x_1(t) &= c_1 + e^{2t}\left[(c_2 + c_3)\cos 3t + (-c_2 + c_3)\sin 3t\right], \\ x_2(t) &= -c_1 + 2e^{2t}(-c_2\cos 3t - c_3\sin 3t), \\ x_3(t) &= 2e^{2t}(c_2\cos 3t + c_3\sin 3t). \end{aligned}$$

26. Characteristic equation $-\lambda^3 + \lambda^2 + 4\lambda + 6 = 0$;
Eigenvalues $\lambda = 3$ and $\lambda = -1 \pm i$;

With $\lambda = 3$ the eigenvector equation $\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 4 & 9 & 0 \end{bmatrix}^{\mathrm{T}}$. With $\lambda = -1 + i$ we solve the eigenvector equation

$$\begin{bmatrix} 4-i & 0 & 1 \\ 9 & i & 2 \\ -9 & 4 & i \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to find the complex-valued eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 & 2-i & -4+i \end{bmatrix}^{\mathrm{T}}$. The corresponding complex-valued solution is

$$\mathbf{x}(t) = \mathbf{v}e^{(-1+i)t} = e^{-t}\begin{bmatrix} \cos t + i\sin t \\ (2\cos t + \sin t) + i(-\cos t + 2\sin t) \\ (-4\cos t - \sin t) + i(\cos t - 4\sin t) \end{bmatrix}$$

with real and imaginary parts $\mathbf{x}_2(t)$ and $\mathbf{x}_3(t)$. Assembling the general solution $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$, we get the scalar equations

$$\begin{aligned} x_1(t) &= 4c_1e^{3t} + e^{-t}(c_2\cos t + c_3\sin t), \\ x_2(t) &= 9c_1e^{3t} + e^{-t}\left[(2c_2 - c_3)\cos t + (c_2 + 2c_3)\sin t\right], \\ x_3(t) &= e^{-t}\left[(-4c_2 + c_3)\cos t + (-c_2 - 4c_3)\sin t\right]. \end{aligned}$$

Finally, the given initial conditions yield the values $c_1 = 1$, $c_2 = -4$, and $c_3 = 1$, so the desired particular solution is

$$\begin{aligned} x_1(t) &= 4e^{3t} - e^{-t}(4\cos t - \sin t), \\ x_2(t) &= 9e^{3t} - e^{-t}(9\cos t + 2\sin t), \\ x_3(t) &= 17e^{-t}\cos t. \end{aligned}$$

27. The coefficient matrix $\mathbf{A}=\begin{bmatrix}-0.2 & 0\\ 0.2 & -0.4\end{bmatrix}$ has characteristic equation $\lambda^2+0.6\lambda+0.08=0$ with eigenvalues $\lambda_1=-0.2$ and $\lambda_2=-0.4$. We find easily that the associated eigenvectors are $\mathbf{v}_1=\begin{bmatrix}1 & 1\end{bmatrix}^T$ and $\mathbf{v}_2=\begin{bmatrix}0 & 1\end{bmatrix}^T$, so we get the general solution

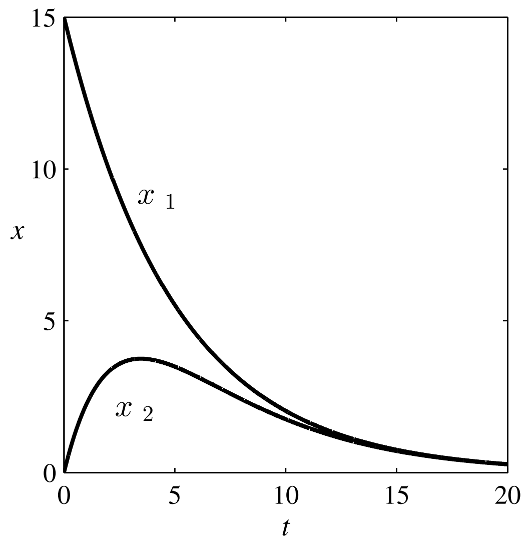
$$x_1(t)=c_1e^{-0.2t},\quad x_2(t)=c_1e^{-0.2t}+c_2e^{-0.4t}.$$

The initial conditions $x_1(0)=15$, $x_2(0)=0$ give $c_1=15$ and $c_2=-15$, so we get

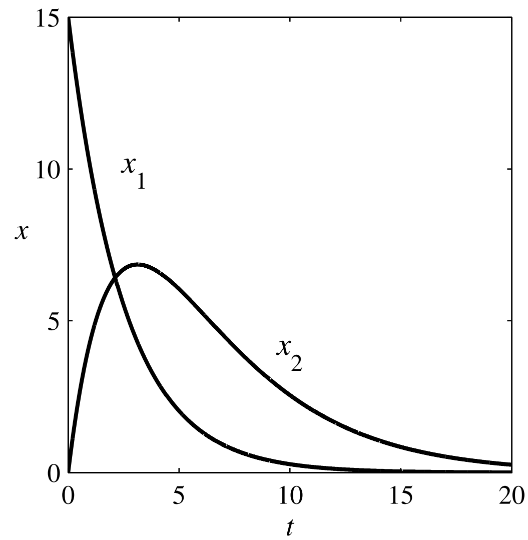
$$x_1(t)=15e^{-0.2t},\quad x_2(t)=15e^{-0.2t}-15e^{-0.4t}.$$

To find the maximum value of $x_2(t)$, we solve the equation $x_2'(t)$ for $t=5\ln 2$, which gives the maximum value $x_2(5\ln 2)=3.75\text{ lb}$. The figure shows the graphs of $x_1(t)$ and $x_2(t)$.

Problem 27



Problem 28



28. The coefficient matrix $\mathbf{A} = \begin{bmatrix}-0.4 & 0\\ 0.4 & -0.25\end{bmatrix}$ has characteristic equation $\lambda^2+0.65\lambda+0.10=0$ with eigenvalues $\lambda_1=-0.4$ and $\lambda_2=-0.25$. We find easily that the associated eigenvectors are $\mathbf{v}_1=\begin{bmatrix}3 & -8\end{bmatrix}^T$ and $\mathbf{v}_2=\begin{bmatrix}0 & 1\end{bmatrix}^T$, so we get the general solution

$$x_1(t)=3c_1e^{-0.4t},\quad x_2(t)=-8c_1e^{-0.4t}+c_2e^{-0.25t}.$$

The initial conditions $x_1(0)=15$, $x_2(0)=0$ give $c_1=5$ and $c_2=40$, so we get

$$x_1(t)=15e^{-0.2t},\quad x_2(t)=-8c_1e^{-0.2t}+c_2e^{-0.4t}\ x_1(t)=15e^{-0.4t},\quad x_2(t)=-40e^{-0.4t}+40e^{-0.25t}.$$

To find the maximum value of $x_2(t)$, we solve the equation $x_2'(t) = 0$ for $t_m = \frac{20}{3}\ln\frac{8}{5}$, which gives the maximum value $x_2(t_m) \approx 6.85$ lb. The figure shows the graphs of $x_1(t)$ and $x_2(t)$.

29. The coefficient matrix $\mathbf{A} = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.6$, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ that yield the general solution

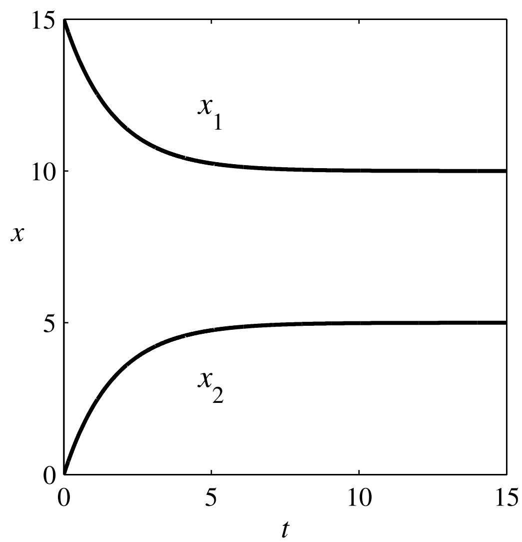
$$x_1(t) = 2c_1 + c_2 e^{-0.6t}, \quad x_2(t) = c_1 - c_2 e^{-0.6t}.$$

The initial conditions $x_1(0) = 15$, $x_2(0) = 0$ give $c_1 = c_2 = 5$, so we get

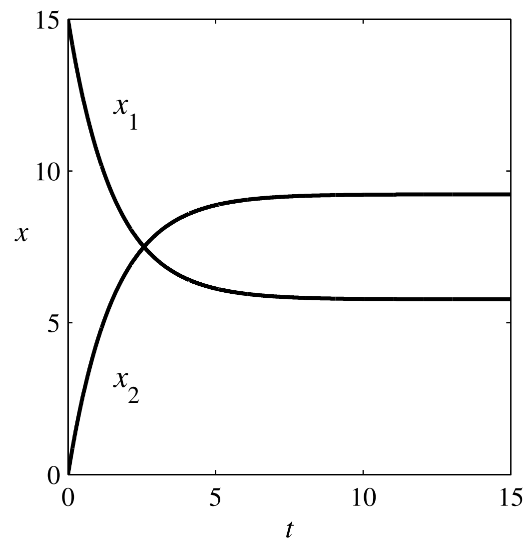
$$x_1(t) = 10 + 5e^{-0.6t}, \quad x_2(t) = 5 - 5e^{-0.6t}.$$

The figure shows the graphs of $x_1(t)$ and $x_2(t)$.

Problem 29



Problem 30



30. The coefficient matrix $\mathbf{A} = \begin{bmatrix} -0.4 & 0.25 \\ 0.4 & -0.25 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.65$, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 5 & 8 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ that yield the general solution

$$x_1(t) = 5c_1 + c_2 e^{-0.65t}, \quad x_2(t) = 8c_1 - c_2 e^{-0.65t}.$$

The initial conditions $x_1(0) = 15$, $x_2(0) = 0$ give $c_1 = \frac{15}{13}$ and $c_2 = \frac{120}{13}$, so we get

$$x_1(t) = \frac{1}{13}\left(75 + 120e^{-0.65t}\right), \quad x_2(t) = \frac{1}{13}\left(120 - 120e^{-0.65t}\right).$$

The figure shows the graphs of $x_1(t)$ and $x_2(t)$.

31. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -3$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^{\mathrm{T}}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$. The resulting general solution is given by

$$x_1(t) = c_1 e^{-t}, \quad x_2(t) = c_1 e^{-t} + c_2 e^{-2t}, \quad x_3(t) = c_1 e^{-t} + 2c_2 e^{-2t} + c_3 e^{-3t}.$$

The initial conditions $x_1(0) = 27$ and $x_2(0) = x_2(0) = 0$ give $c_1 = c_3 = 27$ and $c_2 = -27$, so we get

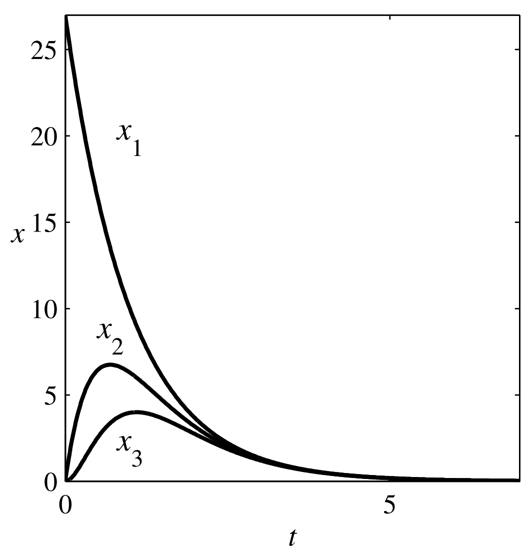
$$x_1(t) = 27e^{-t}, \quad x_2(t) = 27e^{-t} - 27e^{-2t}, \quad x_3(t) = 27e^{-t} - 54e^{-2t} + 27e^{-3t}.$$

The equation $x_3'(t) = 0$ simplifies to the equation

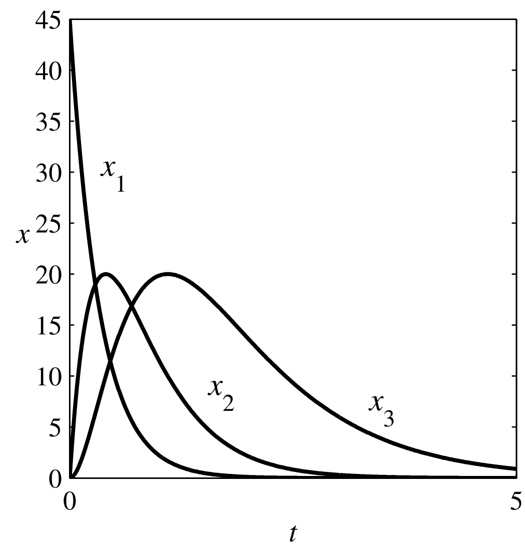
$$3e^{-2t} - 4e^{-t} + 1 = \left(3e^{-t} - 1\right)\left(e^{-t} - 1\right) = 0,$$

with positive solution $t_m = \ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3(\ln 3) = 4$ lb. The figure shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

Problem 31



Problem 32



32. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -3$, $\lambda_2 = -2$, and $\lambda_3 = -1$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 & -3 & 3 \end{bmatrix}^T$, $\mathbf{v}_2 = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix}^T$, and $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The resulting general solution is given by

$$x_1(t) = c_1 e^{-3t}, \quad x_2(t) = -3c_1 e^{-3t} - c_2 e^{-2t}, \quad x_3(t) = 3c_1 e^{-3t} + 2c_2 e^{-2t} + c_3 e^{-t}.$$

The initial conditions $x_1(0) = 45$ and $x_2(0) = x_2(0) = 0$ give $c_1 = 45$, $c_2 = -135$, and $c_3 = 135$, so we get

$$x_1(t) = 45e^{-3t}, \quad x_2(t) = -135e^{-3t} + 135e^{-2t}, \quad x_3(t) = 135e^{-3t} - 270e^{-2t} + 135e^{-t}.$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$3e^{-2t} - 4e^{-t} + 1 = \left(3e^{-t} - 1\right)\left(e^{-t} - 1\right) = 0,$$

with positive solution $t_m = \ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3(\ln 3) = 20 \text{ lb}$. The figure shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

33. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ 4 & -6 & 0 \\ 0 & 6 & -2 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -4$, $\lambda_2 = -6$, and $\lambda_3 = -2$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 & -2 & 6 \end{bmatrix}^T$, $\mathbf{v}_2 = \begin{bmatrix} 0 & -2 & 3 \end{bmatrix}^T$, and $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The resulting general solution is given by

$$x_1(t) = -c_1 e^{-4t}, \quad x_2(t) = -2c_1 e^{-4t} - 2c_2 e^{-6t}, \quad x_3(t) = 6c_1 e^{-4t} + 3c_2 e^{-6t} + c_3 e^{-2t}.$$

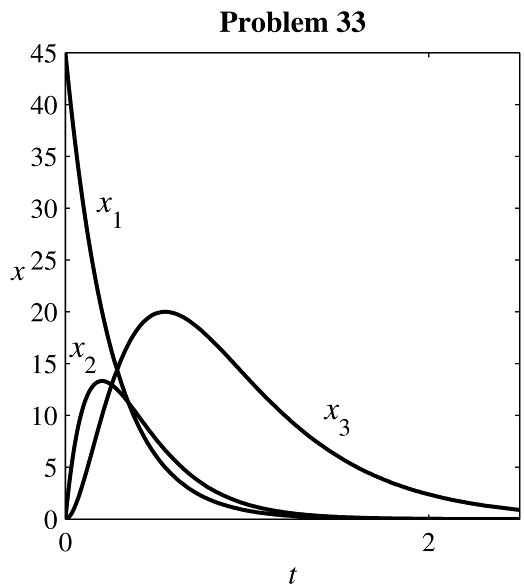
The initial conditions $x_1(0) = 45$ and $x_2(0) = x_2(0) = 0$ give $c_1 = -45$, $c_2 = 45$, and $c_3 = 135$, so we get

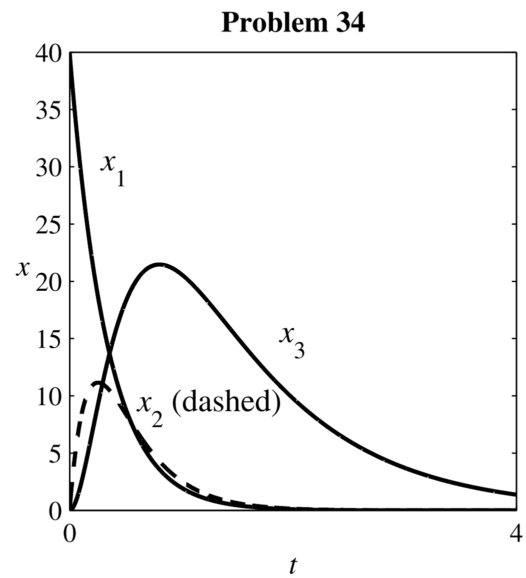
$$x_1(t) = 45e^{-4t}, \quad x_2(t) = 90e^{-4t} - 90e^{-6t}, \quad x_3(t) = -270e^{-4t} + 135e^{-6t} + 135e^{-2t}.$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$3e^{-4t} - 4e^{-2t} + 1 = \left(3e^{-2t} - 1\right)\left(e^{-2t} - 1\right) = 0,$$

with positive solution $t_m = \frac{1}{2}\ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3\left(\frac{1}{2}\ln 3\right) = 20$ lb. The figure shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.





34. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -5 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -3$, $\lambda_2 = -5$, and $\lambda_3 = -1$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -4 & -6 & 15 \end{bmatrix}^T$, $\mathbf{v}_2 = \begin{bmatrix} 0 & -4 & 5 \end{bmatrix}^T$, and $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The resulting general solution is given by

$$x_1(t) = -4c_1e^{-3t}, \quad x_2(t) = -6c_1e^{-3t} - 4c_2e^{-5t}, \quad x_3(t) = 15c_1e^{-3t} + 5c_2e^{-5t} + c_3e^{-t}.$$

The initial conditions $x_1(0) = 40$ and $x_2(0) = x_2(0) = 0$ give $c_1 = -10$, $c_2 = 15$, and $c_3 = 75$, so we get

$$x_1(t) = 40e^{-3t}, \quad x_2(t) = 60e^{-3t} - 60e^{-5t}, \quad x_3(t) = -150e^{-3t} + 75e^{-5t} + 75e^{-t}.$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$5e^{-4t} - 6e^{-2t} + 1 = \left(5e^{-2t} - 1\right)\left(e^{-2t} - 1\right) = 0,$$

with positive solution $t_m = \frac{1}{2}\ln 5$. Thus the maximum amount of salt ever in tank 3 is $x_3\left(\frac{1}{2}\ln 5\right) \approx 21.4663$ lb. The figure shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

35. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -6 & 0 & 3 \\ 6 & -20 & 0 \\ 0 & 20 & -3 \end{bmatrix}$$

has characteristic equation

$$-\lambda^3 - 29\lambda^2 - 198\lambda = -\lambda(\lambda - 18)(\lambda - 11) = 0,$$

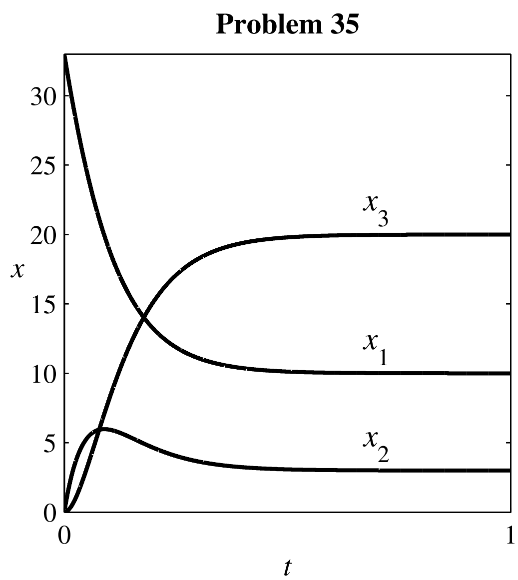
with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -18$, and $\lambda_2 = -11$. We find that associated eigenvectors are $\mathbf{v}_0 = [10 \quad 3 \quad 20]^{\mathrm{T}}$, $\mathbf{v}_1 = [-1 \quad -3 \quad 4]^{\mathrm{T}}$, and $\mathbf{v}_2 = [-3 \quad -2 \quad 5]^{\mathrm{T}}$. The resulting general solution is given by

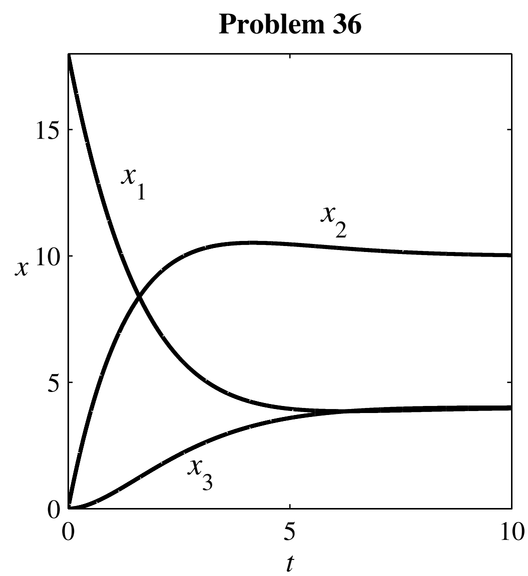
$$\begin{aligned} x_1(t) &= 10c_0 - c_1 e^{-18t} - 3c_2 e^{-11t} \\ x_2(t) &= 3c_0 - 3c_1 e^{-18t} - 2c_2 e^{-11t} \\ x_3(t) &= 20c_0 + 4c_1 e^{-18t} + 5c_2 e^{-11t}. \end{aligned}$$

The initial conditions $x_1(0) = 33$ and $x_2(0) = x_2(0) = 0$ give $c_1 = 1$, $c_2 = \frac{55}{7}$, and $c_3 = -\frac{72}{7}$, so we get

$$\begin{aligned} x_1(t) &= 10 - \frac{1}{7}\left(55e^{-18t} - 216e^{-11t}\right) \\ x_2(t) &= 3 - \frac{1}{7}\left(165e^{-18t} - 144e^{-11t}\right) \\ x_3(t) &= 20 + \frac{1}{7}\left(220e^{-18t} - 360e^{-11t}\right). \end{aligned}$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 10 lb, 3 lb, and 20 lb. The figure shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.





36. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix}$$

has characteristic equation $-\lambda^3 - \frac{6}{5}\lambda^2 - \frac{9}{20}\lambda = 0$, with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -\frac{3}{10}(2+i)$, and $\lambda_2 = -\frac{3}{10}(2-i)$. The eigenvector equation

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1/5 & 0 \\ 0 & 1/5 & -1/2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with the eigenvalue $\lambda_0 = 0$ yields the associated eigenvector $\mathbf{v}_0 = \begin{bmatrix} 1 & 5/2 & 1 \end{bmatrix}^T$ and consequently the constant solution $\mathbf{x}_0(t) \equiv \mathbf{v}_0$. Then the eigenvector equation

$$\cdot \begin{bmatrix} \frac{1}{10}(1+3i) & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{10}(4+3i) & 0 \\ 0 & \frac{1}{5} & \frac{1}{10}(1+3i) \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

associated with $\lambda_1 = -\frac{3}{10}(2+i)$ yields the complex-valued eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{2}(1-3i) & -\frac{1}{2}(1+3i) & 1 \end{bmatrix}^{\mathrm{T}}.$$

The corresponding complex-valued solution is

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{(-6-3i)t/10} = \frac{1}{2}e^{-3t/5}\begin{bmatrix} \left(-\cos\frac{3t}{10} + 3\sin\frac{3t}{10}\right) + i\left(3\cos\frac{3t}{10} + \sin\frac{3t}{10}\right) \\ \left(-\cos\frac{3t}{10} - 3\sin\frac{3t}{10}\right) + i\left(-3\cos\frac{3t}{10} + \sin\frac{3t}{10}\right) \\ 2\cos\frac{3t}{10} - 2i\sin\frac{3t}{10} \end{bmatrix}.$$

The scalar components of the resulting general solution $\mathbf{x} = c_0\mathbf{x}_0 + c_1\operatorname{Re}[\mathbf{x}_1] + c_2\operatorname{Im}[\mathbf{x}_1]$ are given by

$$x_1(t) = c_0 + \tfrac{1}{2}e^{-3t/5}\left[(-c_1 + 3c_2)\cos\frac{3t}{10} + (3c_1 + c_2)\sin\frac{3t}{10}\right]$$

$$x_2(t) = \frac{5}{2}c_0 + \tfrac{1}{2}e^{-3t/5}\left[(-c_1 - 3c_2)\cos\frac{3t}{10} + (-3c_1 + c_2)\sin\frac{3t}{10}\right]$$

$$x_3(t) = c_0 + e^{-3t/5}\left(c_1\cos\frac{3t}{10} - c_2\sin\frac{3t}{10}\right).$$

When we impose the initial conditions $x_1(0) = 18$ and $x_2(0) = x_2(0) = 0$ we find that $c_0 = 4$, $c_1 = -4$, and $c_2 = 8$. This finally gives the particular solution

$$x_1(t) = 4 + e^{-3t/5}\left(14\cos\frac{3t}{10} - 2\sin\frac{3t}{10}\right)$$

$$x_2(t) = 10 - e^{-3t/5}\left(10\cos\frac{3t}{10} - 10\sin\frac{3t}{10}\right)$$

$$x_3(t) = 4 - e^{-3t/5}\left(4\cos\frac{3t}{10} + 8\sin\frac{3t}{10}\right).$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 4 lb, 10 lb, and 4 lb. The figure below shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

37. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix}$$

has characteristic equation $-\lambda^3 - 6\lambda^2 - 11\lambda = 0$, with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -3 - i\sqrt{2}$, and $\lambda_1 = -3 + i\sqrt{2}$. The eigenvector equation

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with the eigenvalue $\lambda_0 = 0$ yields the associated eigenvector $\mathbf{v}_0 = \begin{bmatrix} 6 & 2 & 3 \end{bmatrix}^{\mathrm{T}}$, and consequently the constant solution $\mathbf{x}_0(t) \equiv \mathbf{v}_0$. Then the eigenvector equation

$$\begin{bmatrix} 2 + i\sqrt{2} & 0 & 2 \\ 1 & i\sqrt{2} & 0 \\ 0 & 3 & 1 + i\sqrt{2} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with $\lambda_1 = -3 - i\sqrt{2}$ yields the complex-valued eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{3}\left(-2 + i\sqrt{2}\right) & \frac{1}{3}\left(-1 - i\sqrt{2}\right) & 1 \end{bmatrix}^{\mathrm{T}}.$$

The corresponding complex-valued solution is

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\left(-3 - i\sqrt{2}\right)t} = \frac{1}{3}e^{-3t}\begin{bmatrix} \left[-2\cos\left(\sqrt{2}t\right) + \sqrt{2}\sin\left(\sqrt{2}t\right)\right] + i\left[\sqrt{2}\cos\left(\sqrt{2}t\right) + 2\sin\left(\sqrt{2}t\right)\right] \\ \left[-\cos\left(\sqrt{2}t\right) - \sqrt{2}\sin\left(\sqrt{2}t\right)\right] + i\left[-\sqrt{2}\cos\left(\sqrt{2}t\right) + \sin\left(\sqrt{2}t\right)\right] \\ 3\cos\left(\sqrt{2}t\right) - 3i\sin\left(\sqrt{2}t\right) \end{bmatrix}.$$

The scalar components of resulting general solution $\mathbf{x} = c_0\mathbf{x}_0 + c_1\operatorname{Re}[\mathbf{x}_1] + c_2\operatorname{Im}[\mathbf{x}_1]$ are given by

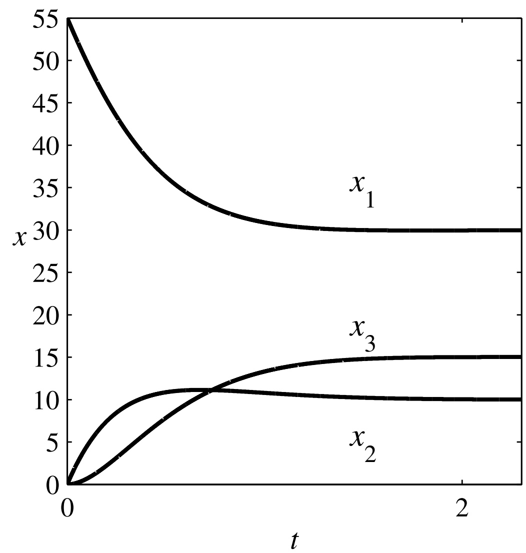
$$\begin{aligned} x_1(t) &= 6c_0 + \frac{1}{3}e^{-3t}\left[\left(-2c_1 + \sqrt{2}c_2\right)\cos\left(\sqrt{2}t\right) + \left(\sqrt{2}c_1 + 2c_2\right)\sin\left(\sqrt{2}t\right)\right] \\ x_2(t) &= 2c_0 + \frac{1}{3}e^{-3t}\left[\left(-c_1 - \sqrt{2}c_2\right)\cos\left(\sqrt{2}t\right) + \left(-\sqrt{2}c_1 + c_2\right)\sin\left(\sqrt{2}t\right)\right] \\ x_3(t) &= 3c_0 + e^{-3t}\left[c_1\cos\left(\sqrt{2}t\right) - c_2\sin\left(\sqrt{2}t\right)\right]. \end{aligned}$$

When we impose the initial conditions $x_1(0) = 55$ and $x_2(0) = x_2(0) = 0$ we find that $c_0 = 5$, $c_1 = -15$, and $c_2 = \dfrac{45}{\sqrt{2}}$. This finally gives the particular solution

$$x_1(t) = 30 + e^{-3t}\left[25\cos\left(\sqrt{2}t\right) + 10\sqrt{2}\sin\left(\sqrt{2}t\right)\right]$$
$$x_2(t) = 10 - e^{-3t}\left[10\cos\left(\sqrt{2}t\right) - \frac{25}{2}\sqrt{2}\sin\left(\sqrt{2}t\right)\right]$$
$$x_3(t) = 15 - e^{-3t}\left[15\cos\left(\sqrt{2}t\right) + \frac{45}{2}\sqrt{2}\sin\left(\sqrt{2}t\right)\right].$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 30 lb, 10 lb, and 15 lb. The figure shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

Problem 37



In Problems 38-41 the Maple command **`with(linalg):eigenvects(A)`**, the *Mathematica* command **`Eigensystem[A]`**, or the MATLAB command **`[V,D] = eig(A)`** can be used to find the eigenvalues and associated eigenvectors of the given coefficient matrix **A**.

38. Characteristic equation: $(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)=0$

Eigenvalues and associated eigenvectors:

λ	1	2	3	4
v	$[1\ \ -2\ \ 3\ \ -4]^T$	$[0\ \ 1\ \ -3\ \ 6]^T$	$[0\ \ 0\ \ 1\ \ -4]^T$	$[0\ \ 0\ \ 0\ \ 1]^T$

Scalar solution equations:

$$x_1(t) = c_1e^{t}$$
$$x_2(t) = -2c_1e^{t} + c_2e^{2t}$$
$$x_3(t) = 3c_1e^{t} - 3c_2e^{2t} + c_3e^{3t}$$
$$x_4(t) = -4c_1e^{t} + 6c_2e^{2t} - 4c_3e^{3t} + c_4e^{4t}$$

39. Characteristic equation: $(\lambda^2-1)(\lambda^2-4)=0$

Eigenvalues and associated eigenvectors:

λ	1	−1	2	−2
v	$[3\ \ -2\ \ 4\ \ 1]^T$	$[0\ \ 0\ \ 1\ \ 0]^T$	$[0\ \ 1\ \ 0\ \ 0]^T$	$[1\ \ -1\ \ 0\ \ 0]^T$

Scalar solution equations:

$$\begin{aligned}
x_1(t) &= 3c_1e^{t} + c_4e^{-2t}\\
x_2(t) &= -2c_1e^{t} + c_3e^{2t} - c_4e^{-2t}\\
x_3(t) &= 4c_1e^{t} + c_2e^{-t}\\
x_4(t) &= c_1e^{t}
\end{aligned}$$

40. Characteristic equation: $(\lambda^2-4)(\lambda^2-25)=0$

Eigenvalues and associated eigenvectors:

λ	2	−2	5	−5
v	$[1\ \ -3\ \ 0\ \ 0]^T$	$[0\ \ 3\ \ 0\ \ -1]^T$	$[0\ \ 0\ \ 1\ \ -3]^T$	$[0\ \ 1\ \ 0\ \ 0]^T$

Scalar solution equations:

$$\begin{aligned}
x_1(t) &= c_1e^{2t}\\
x_2(t) &= -3c_1e^{2t} + 3c_2e^{-2t} - c_4e^{-5t}\\
x_3(t) &= c_3e^{5t}\\
x_4(t) &= -c_2e^{-2t} - 3c_3e^{5t}
\end{aligned}$$

41. The eigenvalues and respective eigenvalues are given by the following table:

λ	−3	−6	10	15
v	$[1\ \ 0\ \ 0\ \ -1]^T$	$[0\ \ 1\ \ -1\ \ 0]^T$	$[-2\ \ 1\ \ 1\ \ -2]^T$	$[1\ \ 2\ \ 2\ \ 1]^T$

Hence the general solution has scalar component functions

$$\begin{aligned}
x_1(t) &= c_1e^{-3t} - 2c_3e^{10t} + c_4e^{15t}\\
x_2(t) &= c_2e^{-6t} + c_3e^{10t} + 2c_4e^{15t}\\
x_3(t) &= -c_2e^{-6t} + c_3e^{10t} + 2c_4e^{15t}\\
x_4(t) &= -c_1e^{-3t} - 2c_3e^{10t} + c_4e^{15t}
\end{aligned}$$

The given initial conditions are satisfied by choosing $c_1=c_2=0$, $c_3=-1$, and $c_4=1$, so the desired particular solution is given by

$$x_1(t)=2e^{10t}+e^{15t}=x_4(t)$$
$$x_2(t)=-e^{10t}+2e^{15t}=x_3(t)$$

In Problems 42–50 we give a general solution in the form $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots$ that exhibits explicitly the eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ of the given coefficient matrix **A**.

42. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}e^{5t}$$

43. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{4t} + c_3\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}e^{8t}$$

44. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}e^{6t} + c_3\begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix}e^{12t}$$

45. $$\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}e^{3t} + c_4\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}e^{6t}$$

46. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}e^{-4t} + c_2\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}e^{4t} + c_4\begin{bmatrix} 3 \\ -2 \\ 3 \\ -3 \end{bmatrix}e^{8t}$$

47. $$\mathbf{x}(t) = c_1\begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}e^{3t} + c_3\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}e^{6t} + c_4\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}e^{9t}$$

48. $$\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}e^{16t} + c_2\begin{bmatrix} 2 \\ 5 \\ 1 \\ -1 \end{bmatrix}e^{32t} + c_3\begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}e^{48t} + c_4\begin{bmatrix} 1 \\ 1 \\ 2 \\ -3 \end{bmatrix}e^{64t}$$

49. $$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 7 \\ 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + c_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} e^{6t} + c_5 \begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \\ 1 \end{bmatrix} e^{9t}$$

50. $$\mathbf{x}(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} e^{-7t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} e^{-4t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} e^{3t} + c_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{5t} + c_5 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{9t} + c_6 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} e^{11t}$$

SECTION 5.3

SOLUTION CURVES OF LINEAR SYSTEMS

This section emphasizes the connection between the algebraic properties of the matrix **A**—specifically, its eigenvalues and eigenvectors—and the characteristic pattern of the phase diagram of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

In Problems 1-16 the eigenvalues, eigenvectors and phase portraits appear in the solutions to Section 5.2. Thus here we simply categorize each phase portrait according to the gallery in Fig. 5.3.16.

1. Saddle point (real eigenvalues of opposite sign)

2. Saddle point (real eigenvalues of opposite sign)

3. Saddle point (real eigenvalues of opposite sign)

4. Saddle point (real eigenvalues of opposite sign)

5. Saddle point (real eigenvalues of opposite sign)

6. Improper nodal source (distinct positive real eigenvalues)

7. Saddle point (real eigenvalues of opposite sign)

8. Center (pure imaginary eigenvalues)

9. Center (pure imaginary eigenvalues)

10. Center (pure imaginary eigenvalues)

11. Spiral source (complex conjugate eigenvalues with positive real part)

12. Spiral source (complex conjugate eigenvalues with positive real part)

13. Spiral source (complex conjugate eigenvalues with positive real part)

14. Spiral source (complex conjugate eigenvalues with positive real part)

15. Spiral source (complex conjugate eigenvalues with positive real part)

16. Improper nodal sink (distinct real eigenvalues)

In Problems 17-28 we "pigeonhole" each phase portrait according to the gallery in Fig. 5.3.16 and give the nature of the eigenvalues of the matrix **A**. Where appropriate we further give approximate values of the corresponding eigenvectors.

17. Center; pure imaginary eigenvalues

18. Improper nodal source; distinct positive real eigenvalues; $\mathbf{v}_1 \approx [0 \quad 1]^T$, $\mathbf{v}_2 \approx [-1 \quad 1]^T$

19. Saddle point; real eigenvalues of opposite sign; $\mathbf{v}_1 \approx [0 \quad 1]^T$ corresponds to the negative eigenvalue and $\mathbf{v}_2 \approx [-1 \quad 1]^T$ to the positive one.

20. Spiral source; complex conjugate eigenvalues with positive real part

21. Proper nodal source; repeated positive real eigenvalue with linearly independent eigenvectors

22. Parallel lines; one zero and one negative real eigenvalue

23. Spiral sink; complex conjugate eigenvalues with negative real part

24. Improper nodal sink; distinct negative real eigenvalues; $\mathbf{v}_1 \approx [1 \quad 1]^T$, $\mathbf{v}_2 \approx [-1 \quad 4]^T$.

25. Saddle point; real eigenvalues of opposite sign; $\mathbf{v}_1 \approx [1 \quad 1]^T$ corresponds to the positive eigenvalue and $\mathbf{v}_2 \approx [4 \quad -1]^T$ to the negative one.

26. Center; pure imaginary eigenvalues

27. Improper nodal source; distinct positive real eigenvalues; $\mathbf{v}_1 \approx \begin{bmatrix} 2 & 3 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_2 \approx \begin{bmatrix} 2 & -1 \end{bmatrix}^{\mathrm{T}}$

28. Spiral sink; complex conjugate eigenvalues with negative real part

29. **a)** If $v_0 = 0$, then $v(t) \equiv 0$ for all *t*, so that the point $(u,v) = \left(u_0 e^{-2t}, 0\right)$ (in oblique coordinates) lies on the *u*-axis. The reverse argument applies if $u_0 = 0$.

b) If both u_0 and v_0 are nonzero, then solving $u = u_0 e^{-2t}$ for *t* gives $t = -\frac{1}{2}\ln\frac{u}{u_0}$, whereas solving $v = v_0 e^{5t}$ for *t* gives $t = \frac{1}{5}\ln\frac{v}{v_0}$. We can thus eliminate *t* to conclude that $-\frac{1}{2}\ln\frac{u}{u_0} = \frac{1}{5}\ln\frac{v}{v_0}$. Then solving for *v* gives $\ln\frac{v}{v_0} = -\frac{5}{2}\ln\frac{u}{u_0}$, or $\frac{v}{v_0} = \exp\left(-\frac{5}{2}\ln\frac{u}{u_0}\right) = \left(\frac{u}{u_0}\right)^{-5/2}$, or finally $v = v_0\left(\frac{u}{u_0}\right)^{-5/2} = v_0 u_0^{5/2} u^{-5/2} = Cu^{-5/2}$, where $C = v_0 u_0^{5/2}$.

30. The chain rule for vector-valued functions, together with the definition of $\tilde{\mathbf{x}}(t)$ as $\mathbf{x}(-t)$, gives

$$\tilde{\mathbf{x}}'(t) = \frac{d}{dt}\left[\tilde{\mathbf{x}}(t)\right] = \frac{d}{dt}\left[\mathbf{x}(-t)\right] = \mathbf{x}'(-t)\cdot\frac{d}{dt}(-t) = \mathbf{x}'(-t)\cdot(-1) = -\mathbf{x}'(-t)$$

for all *t*. Because $\mathbf{x}(t)$ is a solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}'(-t)$ can be replaced with $\mathbf{A}\mathbf{x}(-t)$, which (by definition of $\tilde{\mathbf{x}}$) is $\mathbf{A}\tilde{\mathbf{x}}(t)$. Then the above displayed equation says that $\tilde{\mathbf{x}}'(t) = -\mathbf{A}\tilde{\mathbf{x}}(t)$ for all *t*, which means that $\tilde{\mathbf{x}}(t)$ is a solution of the system $\tilde{\mathbf{x}}' = -\mathbf{A}\tilde{\mathbf{x}}$.

31. If λ is an eigenvalue of **A** with associated eigenvector **v**, then $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$. Taking the negative of both sides of this equation then gives $-(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = -\mathbf{0} = \mathbf{0}$. However, $-(\mathbf{A} - \lambda\mathbf{I})$ can be written as $(-\mathbf{A}) - (-\lambda)\mathbf{I}$, and so $\left[(-\mathbf{A}) - (-\lambda)\mathbf{I}\right]\mathbf{v} = \mathbf{0}$ as well. It follows that $-\lambda$ is an eigenvalue of the matrix $-\mathbf{A}$ with associated eigenvector **v**. This means that if **A** has positive eigenvalues $0 < \lambda_2 < \lambda_1$ with associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then $-\mathbf{A}$ has negative eigenvalues $-\lambda_1 < -\lambda_2 < 0$ associated to these same eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

32. If we suppose that the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has a nonzero constant solution **x**, then $\mathbf{x}' = \frac{d}{dt}\mathbf{x} \equiv 0$ for all *t*, which means that $\mathbf{0} = \mathbf{A}\mathbf{x}$. Hence **x** is a nonzero constant vector with $\mathbf{Ax} = \mathbf{0}$. Conversely, if we assume that there exists a constant vector $\mathbf{x} \neq \mathbf{0}$ with $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}' = \mathbf{0} = \mathbf{Ax}$, so that the system $\mathbf{x}' = \mathbf{Ax}$ has constant solutions other than the zero solution.

33. **a)** Let $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the two linearly independent eigenvectors of **A** associated with the eigenvalue λ, so that $\mathbf{Av}_1 = \lambda\mathbf{v}_1$ and $\mathbf{Av}_2 = \lambda\mathbf{v}_2$. If **v** is any two-dimensional vector, then the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent implies that **v** can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ for some scalars c_1 and c_2. But then the linearity of matrix multiplication gives

$$\mathbf{Av} = \mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\mathbf{Av}_1 + c_2\mathbf{Av}_2 = c_1\lambda\mathbf{v}_1 + c_2\lambda\mathbf{v}_2 = \lambda(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = \lambda\mathbf{v},$$

proving that v is an eigenvector of A.

b) Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Part a) implies that $\mathbf{Av} = \lambda\mathbf{v}$ for all vectors v, so in particular, if we take $\mathbf{v} = [1 \quad 0]^{\mathrm{T}}$, then $\mathbf{Av} = [a \quad c]^{\mathrm{T}} = \lambda\mathbf{v} = [\lambda \quad 0]^{\mathrm{T}}$, proving that $a = \lambda$ and $c = 0$. Similarly, if we take $\mathbf{v} = [0 \quad 1]^{\mathrm{T}}$, then $\mathbf{Av} = [b \quad d]^{\mathrm{T}} = \lambda\mathbf{v} = [0 \quad \lambda]^{\mathrm{T}}$, proving that $b = 0$ and $d = \lambda$. Thus A is given by Eq. (22).

34. Substituting the expressions for $x_1(t)$ and $x_2(t)$ into the expressions for u and v gives

$$u = \frac{2}{\sqrt{5}}\left(4\cos 10t - \cancel{\sin 10t}\right) + \frac{1}{\sqrt{5}}\left(2\cos 10t + \cancel{2\sin 10t}\right) = \frac{10}{\sqrt{5}}\cos 10t = 2\sqrt{5}\cos 10t$$

and

$$v = -\frac{1}{\sqrt{5}}\left(\cancel{4\cos 10t} - \sin 10t\right) + \frac{2}{\sqrt{5}}\left(\cancel{2\cos 10t} + 2\sin 10t\right) = \frac{5}{\sqrt{5}}\sin 10t = \sqrt{5}\sin 10t.$$

35. Write the given equation as $M(x_1, x_2)\,dx_1 + N(x_1, x_2)\,dx_2 = 0$, where $M(x_1, x_2) = 6x_2 - 8x_1$ and $N(x_1, x_2) = 6x_1 - 17x_2$. The equation is exact because $\frac{\partial M}{\partial x_2} = 6 = \frac{\partial N}{\partial x_1}$. Its general solution is therefore given by $F(x_1, x_2) = k$, where $F(x_1, x_2)$ (as discussed in Section 1.6) satisfies the conditions $\frac{\partial F}{\partial x_1} = M$ and $\frac{\partial F}{\partial x_2} = N$ and k is a constant. The first condition implies that

$$F(x_1, x_2) = \int M\,dx_1 = 6x_1x_2 - 4x_1^2 + h(x_2),$$

which specifies F up to the unknown function $h(x_2)$. Then the second condition gives

$$\frac{\partial}{\partial x_2}\left[6x_1x_2 - 4x_1^2 + h(x_2)\right] = 6x_1 + h'(x_2) = 6x_1 - 17x_2,$$

or simply $h'(x_2) = -17x_2$, which means that up to a constant, $h(x_2) = -\frac{17}{2}x_2^2$. Altogether then, the general solution of the given equation is

$$F(x_1, x_2) = 6x_1x_2 - 4x_1^2 - \frac{17}{2}x_2^2 = k.$$

36. Taking $A = -4$, $B = 6$, and $C = -\frac{17}{2}$ gives $B^2 - 4AC = 6^2 - 4\cdot 4\cdot\frac{17}{2} = -100 < 0$. Thus the nontrivial solution curves are indeed elliptical.

37. With the same values of *A*, *B*, and *C* we find that $\frac{B}{A-C} = \frac{6}{-4+\frac{17}{2}} = \frac{6}{9/2} = \frac{4}{3}$. Thus it suffices to confirm that $\theta = \arctan\frac{2}{4}$ satisfies $\tan 2\theta = \frac{4}{3}$. However, this is readily verified using the double-angle formula for the tangent function:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\frac{2}{4}}{1-\left(\frac{2}{4}\right)^2} = \frac{4}{3}.$$

38. **a)** Let $z = r + si$, so that

$$\tilde{\mathbf{v}} = z\cdot\mathbf{v} = (r+si)\cdot\begin{bmatrix}3+5i\\ 4\end{bmatrix}^T = \begin{bmatrix}(r+si)(3+5i)\\ 4(r+si)\end{bmatrix} = \begin{bmatrix}(3r-5s)+i(5r+3s)\\ 4r+4si\end{bmatrix}$$

has real and imaginary parts $\tilde{\mathbf{a}} = [3r-5s \quad 4r]^T$ and $\tilde{\mathbf{b}} = [5r+3s \quad 4s]^T$, respectively. These are perpendicular if and only if

$$\tilde{\mathbf{a}}\cdot\tilde{\mathbf{b}} = (3r-5s)(5r+3s) + 16rs = 0,$$

that is,

$$15r^2 - \cancel{25sr} + \cancel{9rs} - 15s^2 + \cancel{16rs} = 0,$$

or simply $r^2 = s^2$. This is the same as to say that $z = r \pm ir = r(1\pm i)$.

b) If indeed $s = \pm r$, then $\tilde{\mathbf{a}} = [3r-5s \quad 4r]^T$ is either $[-2r \quad 4r]^T$ or $[8r \quad 4r]^T$, each of which is parallel to an axis of the elliptical trajectory shown in Fig. 5.3.12. The same is true of $\tilde{\mathbf{b}} = [5r+3s \quad 4s]^T$, which becomes either $[8r \quad 4r]^T$ or $[2r \quad -4r]^T$ if $s = \pm r$.

39. **a)** The characteristic equation of **A** is given by $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, that is

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

b) The quadratic formula shows that if the solutions to the characteristic equation are pure imaginary, then the coefficient $-(a+d)$ of λ must vanish. Hence the trace $T(\mathbf{A}) = a+d = 0$, which means that $d = -a$. For the same reason, the constant term $ad-bc$ must be positive. Substituting $d=-a$ then gives $ad-bc = -a^2 - bc > 0$, which is impossible if $c = 0$.

40. From the result of Problem 39, the characteristic equation of the matrix $\mathbf{A} = \begin{bmatrix} -5 & 17 \\ -8 & 7 \end{bmatrix}$ is $\lambda^2 - 2\lambda + 101 = 0$, whose roots (the eigenvalues of **A**) are $\lambda = \dfrac{2 \pm \sqrt{4-404}}{2} = 1 \pm 10i$.

An eigenvector $\mathbf{v} = \begin{bmatrix} a & b \end{bmatrix}^T$ of **A** corresponding to $\lambda = 1+10i$ satisfies $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, or

$$\begin{bmatrix} -5-(1+10i) & 17 \\ -8 & 7-(1+10i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} -6-10i & 17 \\ -8 & 6-10i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is equivalent to the system of equations

$$\begin{aligned} (-6-10i)a + 17b &= 0, \\ -8a + (6-10i)b &= 0. \end{aligned}$$

Both of these equations are satisfied if $a = 17$ and $b = 6+10i$, and so we can take $\mathbf{v} = \begin{bmatrix} 17 & 6+10i \end{bmatrix}^T$, with real and imaginary parts $\mathbf{a} = \begin{bmatrix} 17 & 6 \end{bmatrix}^T$ and $\mathbf{b} = \begin{bmatrix} 0 & 10 \end{bmatrix}^T$. Thus by Eq. (5) of this Section, the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 e^t \left(\begin{bmatrix} 17 \\ 6 \end{bmatrix} \cos 10t - \begin{bmatrix} 0 \\ 10 \end{bmatrix} \sin 10t \right) + c_2 e^t \left(\begin{bmatrix} 0 \\ 10 \end{bmatrix} \cos 10t + \begin{bmatrix} 17 \\ 6 \end{bmatrix} \sin 10t \right),$$

or in scalar form,

$$\begin{aligned} x_1(t) &= e^t (17c_1 \cos 10t + 17c_2 \sin 10t), \\ x_2(t) &= e^t \left[(6c_1 + 10c_2)\cos 10t + (6c_2 - 10c_1)\sin 10t \right]. \end{aligned}$$

The initial condition $\mathbf{x}(0) = \begin{bmatrix} 4 & 2 \end{bmatrix}^T$ implies that $17c_1 = 4$ and $6c_1 + 10c_2 = 2$, leading to $c_1 = \dfrac{4}{17}$ and $c_2 = \dfrac{1}{17}$. All told, the solution of the initial value problem in Eq. (59) is

$$x_1(t) = e^t(4\cos 10t + \sin 10t),$$
$$x_2(t) = e^t(2\cos 10t - 2\sin 10t),$$

in agreement with Eq. (61).

SECTION 5.4

SECOND-ORDER SYSTEMS AND MECHANICAL APPLICATIONS

This section uses the eigenvalue method to exhibit realistic applications of linear systems. If a computer system like Maple, *Mathematica*, MATLAB, or even a TI-85/86/89/92/Nspire calculator is available, then a system of more than three railway cars, or a multistory building with four or more floors (as in the project), can be investigated. However, the problems in the text are intended for manual solution.

Problems 1–7 involve the system

$$\begin{aligned} m_1x_1'' &= -(k_1+k_2)x_1 + k_2x_2 \\ m_2x_2'' &= k_2x_1 - (k_2+k_3)x_2 \end{aligned}$$

with various values of m_1, m_2 and k_1, k_2, k_3. In each problem we divide the first equation by m_1 and the second one by m_2 to obtain a second-order linear system $\mathbf{x}'' = \mathbf{Ax}$ in the standard form of Theorem 1 in this section. If the eigenvalues λ_1 and λ_2 are both negative, then the natural (circular) frequencies of the system are $\omega_1 = \sqrt{-\lambda_1}$ and $\omega_2 = \sqrt{-\lambda_2}$, and—according to Eq. (11) in Theorem 1 of this section—the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ associated with λ_1 and λ_2 determine the natural modes of oscillations at these frequencies.

1. The matrix $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ has eigenvalues $\lambda_0 = 0$ and $\lambda_1 = -4$ with associated eigenvectors $\mathbf{v}_0 = [1 \quad 1]^{\mathrm{T}}$ and $\mathbf{v}_1 = [1 \quad -1]^{\mathrm{T}}$. Thus we have the special case described in Eq. (12) of Theorem 1, and a general solution is given by

$$x_1(t) = a_1 + a_2t + b_1\cos 2t + b_2\sin 2t,$$
$$x_2(t) = a_1 + a_2t - b_1\cos 2t - b_2\sin 2t.$$

The natural frequencies are $\omega_1 = 0$ and $\omega_2 = 2$. In the degenerate natural mode with "frequency" $\omega_1 = 0$ the two masses move by translation without oscillating. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

2. The matrix $\mathbf{A} = \begin{bmatrix} -5 & 4 \\ 5 & -5 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -9$ with associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Hence a general solution is given by

$$x_1(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t,$$
$$x_2(t) = a_1 \cos t + a_2 \sin t - b_1 \cos 3t - b_2 \sin 3t.$$

3. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -4$, with associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 2 & -1 \end{bmatrix}^T$. Hence a general solution is given by

$$x_1(t) = a_1 \cos t + a_2 \sin t + 2b_1 \cos 2t + 2b_2 \sin 2t,$$
$$x_2(t) = a_1 \cos t + a_2 \sin t - b_1 \cos 2t - b_2 \sin 2t.$$

The natural frequencies are $\omega_1 = 1$ and $\omega_2 = 2$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. In the natural mode with frequency ω_2 they move in opposite directions with the amplitude of oscillation of m_1 twice that of m_2.

4. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -5$ with associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Hence a general solution is given by

$$x_1(t) = a_1 \cos t + a_2 \sin t + b_1 \cos\left(\sqrt{5}t\right) + b_2 \sin\left(\sqrt{5}t\right),$$
$$x_2(t) = a_1 \cos t + a_2 \sin t - b_1 \cos\left(\sqrt{5}t\right) - b_2 \sin\left(\sqrt{5}t\right).$$

The natural frequencies are $\omega_1 = 1$ and $\omega_2 = \sqrt{5}$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

5. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$ with associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Hence a general solution is given by

$$x_1(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) + b_1 \cos 2t + b_2 \sin 2t,$$
$$x_2(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) - b_1 \cos 2t - b_2 \sin 2t.$$

The natural frequencies are $\omega_1 = \sqrt{2}$ and $\omega_2 = 2$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

6. The matrix $\mathbf{A} = \begin{bmatrix} -6 & 4 \\ 2 & -4 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -8$ with associated eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [2 \quad -1]^T$. Hence a general solution is given by

$$x_1(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) + 2b_1 \cos\left(\sqrt{8}t\right) + 2b_2 \sin\left(\sqrt{8}t\right),$$
$$x_2(t) = a_1 \cos\left(\sqrt{2}t\right) + a_2 \sin\left(\sqrt{2}t\right) - b_1 \cos\left(\sqrt{8}t\right) - b_2 \sin\left(\sqrt{8}t\right).$$

The natural frequencies are $\omega_1 = \sqrt{2}$ and $\omega_2 = \sqrt{8}$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. In the natural mode with frequency ω_2 they move in opposite directions with the amplitude of oscillation of m_1 twice that of m_2.

7. The matrix $\mathbf{A} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}$ has eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -16$ with associated eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$. Hence a general solution is given by

$$x_1(t) = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t,$$
$$x_2(t) = a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t.$$

The natural frequencies are $\omega_1 = 2$ and $\omega_2 = 4$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

8. Substitution of the trial solution $x_1 = c_1 \cos 5t$, $x_2 = c_2 \cos 5t$ in the system

$$x_1'' = -5x_1 + 4x_2 + 96\cos 5t, \quad x_2'' = 4x_1 - 5x_2$$

yields $c_1 = -5$ and $c_2 = 1$, so a general solution is given by

$$x_1(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t - 5\cos 5t,$$
$$x_2(t) = a_1 \cos t + a_2 \sin t - b_1 \cos 3t - b_2 \sin 3t + \cos 5t.$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 2$, $a_2 = 0$, $b_1 = 3$, and $b_2 = 0$. The resulting particular solution is

$$x_1(t) = 2\cos t + 3\cos 3t - 5\cos 5t,$$
$$x_2(t) = 2\cos t - 3\cos 3t + \cos 5t.$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and equal amplitudes;
- in opposite directions with frequency $\omega_2 = 3$ and equal amplitudes;
- in opposite directions with frequency $\omega_3 = 5$ and with the amplitude of motion of m_1 being 5 times that of m_2.

9. Substitution of the trial solution $x_1 = c_1 \cos 3t$, $x_2 = c_2 \cos 3t$ in the system

$$x_1'' = -3x_1 + 2x_2 \quad 2x_2'' = 2x_1 - 4x_2 + 120\cos 3t$$

yields $c_1 = 3$ and $c_2 = -9$, so a general solution is given by

$$x_1(t) = a_1 \cos t + a_2 \sin t + 2b_1 \cos 2t + 2b_2 \sin 2t + 3\cos 3t,$$
$$x_2(t) = a_1 \cos t + a_2 \sin t - b_1 \cos 2t - b_2 \sin 2t - 9\cos 3t.$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 5$, $a_2 = 0$, $b_1 = -4$, and $b_2 = 0$. The resulting particular solution is

$$x_1(t) = 5\cos t - 8\cos 2t + 3\cos 3t,$$
$$x_2(t) = 5\cos t + 4\cos 2t - 9\cos 3t.$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and equal amplitudes;
- in opposite directions with frequency $\omega_2 = 2$ and with the amplitude of motion of m_1 being twice that of m_2;
- in opposite directions with frequency $\omega_3 = 3$ and with the amplitude of motion of m_2 being 3 times that of m_1.

10. Substitution of the trial solution $x_1 = c_1 \cos t$, $x_2 = c_2 \cos t$ in the system

$$x_1'' = -10x_1 + 6x_2 + 30\cos t, \quad x_2'' = 6x_1 - 10x_2 + 60\cos t$$

yields $c_1 = 14$ and $c_2 = 16$, so a general solution is given by

$$x_1(t) = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t + 14\cos t,$$
$$x_2(t) = a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t + 16\cos t.$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 1$ $a_2 = 0$, $b_1 = -15$, and $b_2 = 0$. The resulting particular solution is

$$x_1(t) = \cos 2t - 15\cos 4t + 14\cos t,$$
$$x_2(t) = \cos 2t + 15\cos 4t + 16\cos t.$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and with the amplitude of motion of m_2 being $\frac{8}{7}$ times that of m_2;
- in the same direction with frequency $\omega_2 = 2$ and equal amplitudes;
- in opposite directions with frequency $\omega_3 = 4$ and equal amplitudes.

11. **(a)** The matrix $\mathbf{A} = \begin{bmatrix} -40 & 8 \\ 12 & -60 \end{bmatrix}$ has eigenvalues $\lambda_1 = -36$ and $\lambda_2 = -64$ with associated eigenvalues $\mathbf{v}_1 = [2 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -3]^T$. Hence a general solution is given by

$$x(t) = 2a_1 \cos 6t + 2a_2 \sin 6t + b_1 \cos 8t + b_2 \sin 8t,$$
$$y(t) = a_1 \cos 6t + a_2 \sin 6t - 3b_1 \cos 8t - 3b_2 \sin 8t.$$

The natural frequencies are $\omega_1 = 6$ and $\omega_2 = 8$. In mode 1 the two masses oscillate in the same direction with frequency $\omega_1 = 6$ and with the amplitude of motion of m_1 being twice that of m_2. In mode 2 the two masses oscillate in opposite directions with frequency $\omega_2 = 8$ and with the amplitude of motion of m_2 being 3 times that of m_1.

(b) Substitution of the trial solution $x = c_1 \cos 7t$, $y = c_2 \cos 7t$ in the system

$$x'' = -40x + 8y - 195\cos 7t, \quad y'' = 12x - 60y - 195\cos 7t$$

yields $c_1 = 19$ and $c_2 = 3$, so a general solution is given by

$$x(t) = 2a_1 \cos 6t + 2a_2 \sin 6t + b_1 \cos 8t + b_2 \sin 8t + 19\cos 7t,$$
$$y(t) = a_1 \cos 6t + a_2 \sin 6t - 3b_1 \cos 8t - 3b_2 \sin 8t + 3\cos 7t.$$

Imposition of the initial conditions $x(0) = 19$, $x'(0) = 12$, $y(0) = 3$, and $y'(0) = 6$ now yields $a_1 = 0$, $a_2 = 1$, $b_1 = 0$, and $b_2 = 0$. The resulting particular solution is

$$x(t) = 2\sin 6t + 19\cos 7t,$$
$$y(t) = \sin 6t + 3\cos 7t.$$

Thus the expected oscillation with frequency $\omega_2 = 8$ is missing, and we have a superposition of (only two) oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 6$ and with the amplitude of motion of m_1 being twice that of m_2;

- in the same direction with frequency $\omega_3 = 7$ and with the amplitude of motion of m_1 being $\frac{19}{3}$ times that of m_2.

12. The coefficient matrix $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ has characteristic polynomial

$$\lambda^3 + 6\lambda^2 + 10\lambda + 4 = (\lambda + 2)(\lambda^2 + 4\lambda + 2).$$

Its eigenvalues $\lambda_1 = -2$, $\lambda_2 = -2 - \sqrt{2}$, and $\lambda_3 = -2 + \sqrt{2}$ have associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, $\mathbf{v}_2 = \begin{bmatrix} 1 & -\sqrt{2} & 1 \end{bmatrix}^T$, and $\mathbf{v}_3 = \begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}^T$. Hence the system's three natural modes of oscillation have

- Natural frequency $\omega_1 = \sqrt{2}$ with amplitude ratios $1:0:-1$;
- Natural frequency $\omega_2 = \sqrt{2+\sqrt{2}}$ with amplitude ratios $1:-\sqrt{2}:1$.
- Natural frequency $\omega_3 = \sqrt{2-\sqrt{2}}$ with amplitude ratios $1:\sqrt{2}:1$.

13. The coefficient matrix $\mathbf{A} = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -4 \end{bmatrix}$ has characteristic polynomial

$$-\lambda^3 - 12\lambda^2 - 40\lambda - 32 = -(\lambda + 4)(\lambda^2 + 8\lambda + 8).$$

Its eigenvalues $\lambda_1 = -4$, $\lambda_2 = -4 - 2\sqrt{2}$, and $\lambda_3 = -4 + 2\sqrt{2}$ have associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, $\mathbf{v}_2 = \begin{bmatrix} 1 & -\sqrt{2} & 1 \end{bmatrix}^T$, and $\mathbf{v}_3 = \begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}^T$. Hence the system's three natural modes of oscillation have

- Natural frequency $\omega_1 = 2$ with amplitude ratios $1:0:-1$;
- Natural frequency $\omega_2 = \sqrt{4+2\sqrt{2}}$ with amplitude ratios $1:-\sqrt{2}:1$.
- Natural frequency $\omega_3 = \sqrt{4-2\sqrt{2}}$ with amplitude ratios $1:\sqrt{2}:1$.

14. The equations of motion of the given system are

$$\begin{aligned} x_1'' &= -50x_1 + 10(x_2 - x_1) + 5\cos 10t, \\ m_2 x_2'' &= -10(x_2 - x_1). \end{aligned}$$

When we substitute $x_1 = A\cos 10t$, $x_2 = B\cos 10t$ and cancel $\cos 10t$ throughout we get the equations

$$\begin{aligned} -40A \quad &- \quad 10B &= \quad 5, \\ -10A \quad &+ \quad (10 - 100m_2)B &= \quad 0. \end{aligned}$$

If $m_2 = 0.1$ (slug), then it follows that $A = 0$, so the mass m_1 remains at rest.

15. First we need the general solution of the homogeneous system $\mathbf{x}'' = \mathbf{Ax}$ with

$$\mathbf{A} = \begin{bmatrix} -50 & 25/2 \\ 50 & -50 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -25$ and $\lambda_2 = -75$, so the natural frequencies of the system are $\omega_1 = 5$ and $\omega_2 = 5\sqrt{3}$. The associated eigenvectors are $\mathbf{v}_1 = [1 \quad 2]^T$ and $\mathbf{v}_2 = [1 \quad -2]^T$, so the complementary solution $\mathbf{x}_c(t)$ is given by

$$\begin{aligned} x_1(t) &= a_1 \cos 5t + a_2 \sin 5t + b_1 \cos\left(5\sqrt{3}t\right) + b_2 \sin\left(5\sqrt{3}t\right), \\ x_2(t) &= 2a_1 \cos 5t + 2a_2 \sin 5t - 2b_1 \cos\left(5\sqrt{3}t\right) - 2b_2 \sin\left(5\sqrt{3}t\right). \end{aligned}$$

When we substitute the trial solution $\mathbf{x}_p(t) = [c_1 \quad c_2]^T \cos 10t$ in the nonhomogeneous system, we find that $c_1 = \frac{4}{3}$ and $c_2 = -\frac{16}{3}$, so a particular solution $\mathbf{x}_p(t)$ is described by

$$x_1(t) = \frac{4}{3}\cos 10t, \quad x_2(t) = -\frac{16}{3}\cos 10t.$$

Finally, when we impose the zero initial conditions on the solution $\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$ we find that $a_1 = \frac{2}{3}$, $a_2 = 0$, $b_1 = -2$, and $b_2 = 0$. Thus the solution we seek is described by

$$\begin{aligned} x_1(t) &= \frac{2}{3}\cos 5t - 2\cos\left(5\sqrt{3}t\right) + \frac{4}{3}\cos 10t, \\ x_2(t) &= \frac{4}{3}\cos 5t + 4\cos\left(5\sqrt{3}t\right) + \frac{16}{3}\cos 10t \end{aligned}$$

We have a superposition of two oscillations with the natural frequencies $\omega_1 = 5$ and $\omega_2 = 5\sqrt{3}$ and a forced oscillation with frequency $\omega = 10$. In each of the two natural oscillations the amplitude of motion of m_2 is twice that of m_1, while in the forced oscillation the amplitude of motion of m_2 is four times that of m_1.

16. The characteristic equation of **A** is

$$(-c_1 - \lambda)(-c_2 - \lambda) - c_1 c_2 = \lambda^2 + (c_1 + c_2)\lambda = 0,$$

whence the given eigenvalues and eigenvectors follow readily.

17. With $c_1 = c_2 = 2$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 2$, $\mathbf{v}_2 = [1 \quad -1]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1 t + a_2 \cos 2t + b_2 \sin 2t,$$
$$x_2(t) = a_1 + b_1 t - a_2 \cos 2t - b_2 \sin 2t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'(0) = 0$ yield $a_1 = a_2 = 0$, $b_1 = \frac{v_0}{2}$, and $b_2 = \frac{v_0}{4}$, so

$$x_1(t) = \frac{v_0}{4}(2t + \sin 2t), \quad x_2(t) = \frac{v_0}{4}(2t - \sin 2t),$$

while $x_2 - x_1 = \frac{v_0}{4}(-2\sin 2t) < 0$; that is, until $t = \frac{\pi}{2}$. Finally, $x_1'\left(\frac{\pi}{2}\right) = 0$ and $x_2'\left(\frac{\pi}{2}\right) = v_0$.

18. With $c_1 = 6$ and $c_2 = 3$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 3$, $\mathbf{v}_2 = [2 \quad -1]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1 t + 2a_2 \cos 3t + 2b_2 \sin 3t,$$
$$x_2(t) = a_1 + b_1 t - a_2 \cos 3t - b_2 \sin 3t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'(0) = 0$ yield $a_1 = a_2 = 0$, $b_1 = \frac{v_0}{3}$, and $b_2 = \frac{v_0}{9}$, so

$$x_1(t) = \frac{v_0}{9}(3t + 2\sin 3t), \quad x_2(t) = \frac{v_0}{9}(3t - \sin 3t),$$

while $x_2 - x_1 = \frac{v_0}{9}(-3\sin 3t) < 0$; that is, until $t = \frac{\pi}{3}$. Finally, $x_1'\left(\frac{\pi}{3}\right) = -\frac{v_0}{3}$ and $x_2'\left(\frac{\pi}{3}\right) = \frac{2v_0}{3}$.

19. With $c_1 = 1$ and $c_2 = 3$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 2$, $\mathbf{v}_2 = [1 \quad -3]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1 t + a_2 \cos 2t + b_2 \sin 2t,$$
$$x_2(t) = a_1 + b_1 t - 3a_2 \cos 2t - 3b_2 \sin 2t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'(0) = 0$ yield $a_1 = a_2 = 0$, $b_1 = \frac{3v_0}{4}$, and $b_2 = \frac{v_0}{8}$, so

$$x_1(t) = \frac{v_0}{8}(6t + \sin 2t), \quad x_2(t) = \frac{v_0}{8}(6t - 3\sin 2t),$$

while $x_2 - x_1 = \frac{v_0}{8}(-4\sin 2t) < 0$; that is, until $t = \frac{\pi}{2}$. Finally, $x_1'\left(\frac{\pi}{2}\right) = \frac{v_0}{2}$ and $x_2'\left(\frac{\pi}{2}\right) = \frac{3v_0}{2}$.

The method of solution in each of Problems 20–23 is the same as that in Example 2 in this section. Thus, looking at the equations in (26), we need to solve the equations

$$\begin{aligned} b_1 + 2b_2 + 4b_3 &= x_1'(0) \\ b_1 - 12b_3 &= x_2'(0) \\ b_1 - 2b_2 + 4b_3 &= x_3'(0) \end{aligned}$$

for the coefficients b_1, b_2, b_3 after inserting given initial values $x_1'(0)$, $x_2'(0)$, $x_3'(0)$ of the three railway cars.

20. With $x_1'(0) = v_0$, $x_2'(0) = 0$, and $x_3'(0) = -v_0$, substitution of the resulting coefficient values b_1, b_2, b_3 in (25) gives the railway car displacement functions

$$x_1(t) = \frac{1}{2}v_0 \sin 2t, \quad x_2(t) = 0, \quad x_3(t) = -\frac{1}{2}v_0 \sin 2t$$

and their velocities

$$x_1'(t) = v_0 \cos 2t, \quad x_2(t) = 0, \quad x_3(t) = -v_0 \cos 2t.$$

We then see that

$$x_2(t) - x_1(t) = x_3(t) - x_2(t) = -\frac{1}{2}v_0 \sin 2t$$

remains negative until $t = \frac{\pi}{2}$, at which time the cars separate with velocities

$$x_1'\left(\frac{\pi}{2}\right) = -v_0, \quad x_2'\left(\frac{\pi}{2}\right) = 0, \quad x_3'\left(\frac{\pi}{2}\right) = v_0.$$

Thus the car in the center remains fixed thereafter and the first and third cars rebound from the collision with the same speeds with which they approached it.

21. With $x_1'(0) = 2v_0$, $x_2'(0) = 0$, and $x_3'(0) = -v_0$, substitution of the resulting coefficient values b_1, b_2, b_3 in (25) gives the railway car displacement functions

$$x_1(t) = \frac{1}{32}v_0(12t + 24\sin 2t + \sin 4t),$$
$$x_2(t) = \frac{1}{32}v_0(12t \qquad\qquad - 3\sin 4t),$$
$$x_3(t) = \frac{1}{32}v_0(12t - 24\sin 2t + \sin 4t).$$

We then see (substituting $\sin 4t = 2\sin 2t\cos 2t$) that

$$x_2(t) - x_1(t) = -\frac{1}{8}v_0(\sin 4t + 6\sin 2t) = -\frac{1}{4}v_0(\sin 2t)(\cos 2t + 3)$$

remains negative until $t = \frac{\pi}{2}$ (as does $x_3(t) - x_2(t)$, similarly) at which time the cars separate with velocities

$$x_1'\left(\frac{\pi}{2}\right) = -v_0, \quad x_2'\left(\frac{\pi}{2}\right) = 0, \quad x_3'\left(\frac{\pi}{2}\right) = 2v_0 .$$

Thus the car in the center remains fixed thereafter, whereas the first and third cars rebound in opposite directions, having exchanged their original velocities.

22. With $x_1'(0) = v_0$, $x_2'(0) = v_0$, and $x_3'(0) = -2v_0$, substitution of the resulting coefficient values b_1, b_2, b_3 in (25) gives the railway car displacement functions

$$x_1(t) = \frac{1}{32}v_0(-4t + 24\sin 2t - 3\sin 4t),$$
$$x_2(t) = \frac{1}{32}v_0(-4t \qquad\qquad + 9\sin 4t),$$
$$x_3(t) = \frac{1}{32}v_0(-4t - 24\sin 2t - 3\sin 4t).$$

We then see (substituting $\sin 4t = 2\sin 2t\cos 2t$) that

$$x_2(t) - x_1(t) = -\frac{3}{8}v_0(2\sin 2t - \sin 4t) = -\frac{3}{4}v_0(\sin 2t)(1 - \cos 2t)$$

remains negative until $t = \frac{\pi}{2}$ (as does $x_3(t) - x_2(t)$, similarly) at which time the cars separate with velocities

$$x_1'\left(\frac{\pi}{2}\right) = -2v_0, \quad x_2'\left(\frac{\pi}{2}\right) = v_0, \quad x_3'\left(\frac{\pi}{2}\right) = v_0 .$$

Thus the car in the center proceeds thereafter with the same velocity it had originally, whereas the first and third cars rebound in opposite directions, having exchanged their original velocities.

23. With $x_1'(0) = 3v_0$, $x_2'(0) = 2v_0$, and $x_3'(0) = 2v_0$, substitution of the resulting coefficient values b_1, b_2, b_3 in (25) gives the railway car displacement functions

$$x_1(t) = \frac{1}{32}v_0(76t + 8\sin 2t + \sin 4t),$$
$$x_2(t) = \frac{1}{32}v_0(76t \qquad - 3\sin 4t),$$
$$x_3(t) = \frac{1}{32}v_0(76t - 8\sin 2t + \sin 4t).$$

We then see (substituting $\sin 4t = 2\sin 2t\cos 2t$) that

$$x_2(t) - x_1(t) = -\frac{1}{8}v_0(2\sin 2t + \sin 4t) = -\frac{1}{4}v_0(\sin 2t)(1 + \cos 2t)$$

remains negative until $t = \frac{\pi}{2}$ (as does $x_3(t) - x_2(t)$, similarly) at which time the cars separate with velocities

$$x_1'\left(\frac{\pi}{2}\right) = 2v_0, \quad x_2'\left(\frac{\pi}{2}\right) = 2v_0, \quad x_3'\left(\frac{\pi}{2}\right) = 3v_0 .$$

Thus the car in the center proceeds thereafter with the same velocity it had originally, whereas the first and third cars rebound in opposite directions, having exchanged their original velocities.

24. With $c_1 = c_3 = 4$ and $c_2 = 16$ the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 4 & 0 \\ 16 & -32 & 16 \\ 0 & 4 & -4 \end{bmatrix}$$

is

$$\lambda^3 + 40\lambda^2 + 144\lambda = \lambda(\lambda + 4)(\lambda + 36) = 0 .$$

The resulting eigenvalues, natural frequencies, and associated eigenvectors are

$$\lambda_1 = 0, \qquad \omega_1 = 0, \quad \mathbf{v}_1 = [1 \quad 1 \quad 1]^T,$$
$$\lambda_2 = 4, \qquad \omega_2 = 2, \quad \mathbf{v}_2 = [1 \quad 0 \quad -1]^T,$$
$$\lambda_3 = -36, \quad \omega_3 = 6, \quad \mathbf{v}_3 = [1 \quad -8 \quad 1]^T.$$

Theorem 1 then gives the general solution

$$\begin{aligned}
x_1(t) &= a_1 + b_1 t + a_2\cos 2t + b_2\sin 2t + a_3\cos 6t + b_3\sin 6t,\\
x_2(t) &= a_1 + b_1 t - 8a_3\cos 6t - 8b_3\sin 6t,\\
x_3(t) &= a_1 + b_1 t - a_2\cos 2t - b_2\sin 2t + a_3\cos 6t + b_3\sin 6t.
\end{aligned}$$

The initial conditions yield $a_1 = a_2 = a_3 = 0$, $b_1 = \frac{4v_0}{9}$, $b_2 = \frac{v_0}{4}$, and $b_3 = \frac{v_0}{108}$, so

$$\begin{aligned}
x_1(t) &= \frac{v_0}{108}(48t + 27\sin 2t + \sin 6t),\\
x_2(t) &= \frac{v_0}{108}(48t - 8\sin 6t),\\
x_3(t) &= \frac{v_0}{108}(48t - 27\sin 2t + \sin 6t),
\end{aligned}$$

while

$$x_2 - x_1 = -18\sin 2t\left(3 - 2\sin^3 2t\right) < 0, \quad x_3 - x_2 = -9\left(4\sin^3 2t\right) < 0,$$

that is, until $t = \frac{\pi}{2}$. Finally

$$x_1'\left(\frac{\pi}{2}\right) = -\frac{v_0}{9}, \quad x_2'\left(\frac{\pi}{2}\right) = \frac{8v_0}{9}, \quad x_3'\left(\frac{\pi}{2}\right) = \frac{8v_0}{9}.$$

25. **(a)** The matrix

$$\mathbf{A} = \begin{bmatrix} -160/3 & 320/3 \\ 8 & -116 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -41.8285$ and $\lambda_2 = -127.5049$, so the natural frequencies are

$$\omega_1 \approx 6.4675\,\text{rad/sec} \approx 1.0293\,\text{Hz}, \quad \omega_2 \approx 11.2918\,\text{rad/sec} \approx 1.7971\,\text{Hz}.$$

(b) Resonance occurs at the two critical speeds

$$v_1 = \frac{20\omega_1}{\pi} \approx 41\,\text{ft/sec} \approx 28\,\text{mi/h}, \quad v_2 = \frac{20\omega_2}{\pi} \approx 72\,\text{ft/sec} \approx 49\,\text{mi/h}.$$

26. With $k_1 = k_2 = k$ and $L_1 = L_2 = \frac{L}{2}$ the equations in (42) reduce to

$$mx'' = -2kx \quad \text{and} \quad I\theta'' = -\frac{kL^2}{2}\theta.$$

The first equation yields $\omega_1 = \sqrt{\frac{2k}{m}}$ and the second one yields $\omega_2 = \sqrt{\frac{kL^2}{2I}}$.

In Problems 27–29 we substitute the given physical parameters into the equations in (42):

$$\begin{aligned} mx'' &= -(k_1+k_2)x + (k_1L_1 - k_2L_2)\theta, \\ I\theta'' &= (k_1L_1 - k_2L_2)x - (k_1L_1^2 + k_2L_2^2)\theta. \end{aligned}$$

As in Problem 25, a critical frequency of ω rad/sec yields a critical velocity of $v = \dfrac{20\omega}{\pi}$ ft/sec .

27. $100x'' = -4000x$, $800\theta'' = 100000\theta$.

Obviously the matrix $\mathbf{A} = \begin{bmatrix} -40 & 0 \\ 0 & -125 \end{bmatrix}$ has eigenvalues $\lambda_1 = -40$ and $\lambda_2 = -125$.

Up-and-down: $\omega_1 = \sqrt{40}$, $v_1 \approx 40.26$ ft/sec ≈ 27 mi/h ;

Angular: $\omega_2 = \sqrt{125}$, $v_2 \approx 71.18$ ft/sec ≈ 49 mi/h .

28. $100x'' = -4000x + 4000\theta$, $1000\theta'' = 4000x - 104000\theta$.

The matrix $\mathbf{A} = \begin{bmatrix} -40 & 40 \\ 4 & -104 \end{bmatrix}$ has eigenvalues $\lambda_1, \lambda_2 = 4\left(-18 \pm \sqrt{74}\right)$.

$\omega_1 \approx 6.1311$, $v_1 \approx 39.03$ ft/sec ≈ 27 mi/h ;

$\omega_2 \approx 10.3155$, $v_1 \approx 65.67$ ft/sec ≈ 45 mi/h .

29. $100x'' = -3000x - 5000\theta$, $800\theta'' = -5000x - 75000\theta$.

The matrix $\mathbf{A} = \begin{bmatrix} -30 & -50 \\ -25/4 & -375/4 \end{bmatrix}$ has eigenvalues $\lambda_1, \lambda_2 = \dfrac{5}{8}\left(-99 \pm \sqrt{3401}\right)$.

$\omega_1 \approx 5.0424$, $v_1 \approx 32.10$ ft/sec ≈ 22 mi/h ;

$\omega_2 \approx 9.9158$, $v_2 \approx 63.13$ ft/sec ≈ 43 mi/h .

SECTION 5.5

MULTIPLE EIGENVALUE SOLUTIONS

In each of Problems 1–6 we give first the characteristic equation with repeated (multiplicity 2) eigenvalue λ. In each case we find that $(\mathbf{A} - \lambda\mathbf{I})^2 = \mathbf{0}$. Then $\mathbf{w} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is a generalized eigenvector and $\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} \neq \mathbf{0}$ is an ordinary eigenvector associated with λ. We give finally the scalar component functions $x_1(t)$, $x_2(t)$ of the general solution

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2(\mathbf{v}t + \mathbf{w})e^{\lambda t}$$

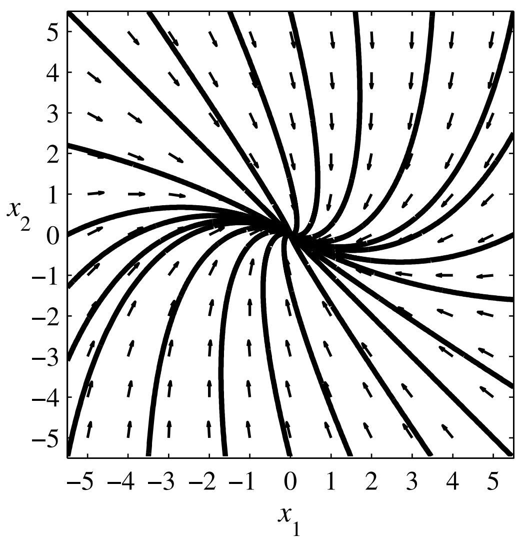
of the given system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

1. Characteristic equation $\lambda^2 + 6\lambda + 9 = 0$; repeated eigenvalue $\lambda = -3$; generalized eigenvector $\mathbf{w} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$;

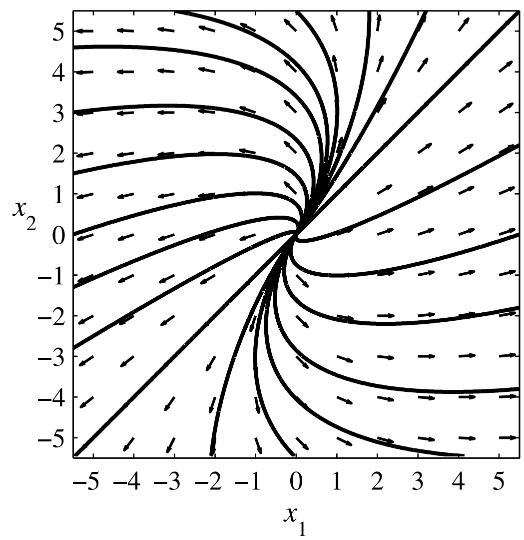
$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix};$$

$$x_1(t) = (c_1 + c_2 + c_2 t)e^{-3t}, \quad x_2(t) = (-c_1 - c_2 t)e^{-3t}.$$

Problem 1



Problem 2



2. Characteristic equation $\lambda^2 - 4\lambda + 4 = 0$; repeated eigenvalue $\lambda = 2$; generalized eigenvector $\mathbf{w} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$;

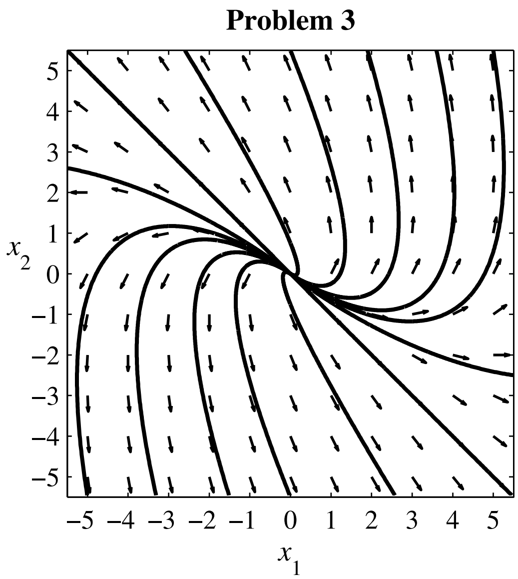
$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

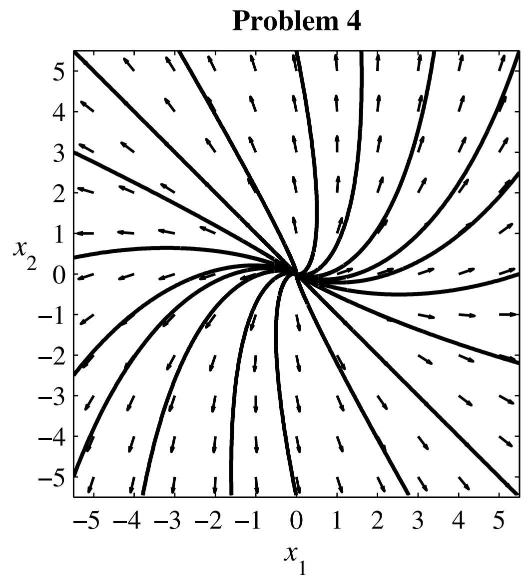
$$x_1(t) = (c_1 + c_2 + c_2 t)e^{2t}, \quad x_2(t) = (c_1 + c_2 t)e^{2t}.$$

3. Characteristic equation $\lambda^2 - 6\lambda + 9 = 0$; repeated eigenvalue $\lambda = 3$; generalized eigenvector $\mathbf{w} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$;

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix};$$

$$x_1(t) = (-2c_1 + c_2 - 2c_2 t)e^{3t}, \quad x_2(t) = (2c_1 + 2c_2 t)e^{3t}.$$





4. Characteristic equation $\lambda^2 - 8\lambda + 16 = 0$; repeated eigenvalue $\lambda = 4$; generalized eigenvector $\mathbf{w} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$;

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix};$$

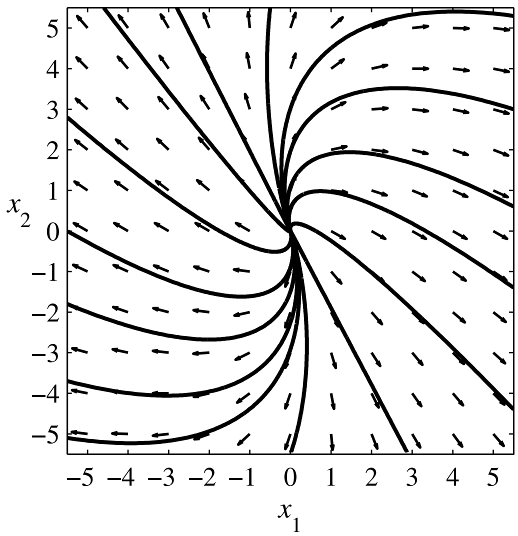
$$x_1(t) = (-c_1 + c_2 - c_2 t)e^{4t}, \quad x_2(t) = (c_1 + c_2 t)e^{4t}.$$

5. Characteristic equation $\lambda^2 - 10\lambda + 25 = 0$; repeated eigenvalue $\lambda = 5$; generalized eigenvector $\mathbf{w} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$;

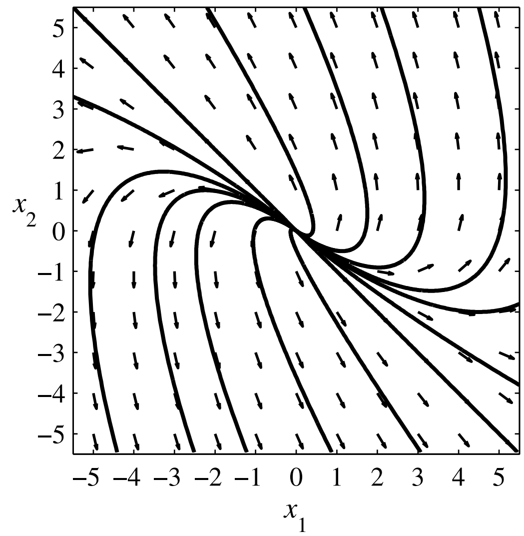
$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix};$$

$$x_1(t) = (2c_1 + c_2 + 2c_2 t)e^{5t}, \quad x_2(t) = (-4c_1 - 4c_2 t)e^{5t}.$$

Problem 5



Problem 6



6. Characteristic equation $\lambda^2 - 10\lambda + 25 = 0$; repeated eigenvalue $\lambda = 5$; generalized eigenvector $\mathbf{w} = [1 \quad 0]^{\mathrm{T}}$;

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{w} = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix};$$

$$x_1(t) = (-4c_1 + c_2 - 4c_2 t)\,e^{5t}, \quad x_2(t) = (4c_1 + 4c_2 t)\,e^{5t}.$$

In each of Problems 7–10 the characteristic polynomial is easily calculated by expansion along the row or column of **A** that contains two zeros. The matrix **A** has only two distinct eigenvalues, so we write $\lambda_1, \lambda_2, \lambda_3$ with either $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$. Nevertheless, we find that it has 3 linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. We list also the scalar components $x_1(t), x_2(t), x_3(t)$ of the general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$ of the system.

7. Characteristic equation $-\lambda^3 + 13\lambda^2 - 40\lambda + 36 = -(\lambda - 2)^2(\lambda - 9)$;

Eigenvalues $\lambda = 2, 2, 9$;

Eigenvectors $[1 \quad 1 \quad 0]^{\mathrm{T}}$, $[1 \quad 0 \quad 1]^{\mathrm{T}}$, $[0 \quad 1 \quad 0]^{\mathrm{T}}$;

$$\begin{aligned} x_1(t) &= c_1 e^{2t} + c_2 e^{2t} \\ x_2(t) &= c_1 e^{2t} + c_3 e^{9t} \\ x_3(t) &= c_2 e^{2t} \end{aligned}$$

8. Characteristic equation $-\lambda^3 + 33\lambda^2 - 351\lambda + 1183 = -(\lambda - 13)^2(\lambda - 7)$;

Eigenvalues $\lambda = 7, 13, 13$;

Eigenvectors $\begin{bmatrix}2 & -3 & 1\end{bmatrix}^T$, $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$, $\begin{bmatrix}-1 & 1 & 0\end{bmatrix}^T$;

$$\begin{aligned}x_1(t) &= 2c_1e^{7t} - c_3e^{13t}\\ x_2(t) &= -3c_1e^{7t} + c_3e^{13t}\\ x_3(t) &= c_1e^{7t} + c_2e^{13t}\end{aligned}$$

9. Characteristic equation $-\lambda^3+19\lambda^2-115\lambda+225=-(\lambda-5)^2(\lambda-9)$;

Eigenvalues $\lambda=5,5,9$;

Eigenvectors $\begin{bmatrix}1 & 2 & 0\end{bmatrix}^T$, $\begin{bmatrix}7 & 0 & 2\end{bmatrix}^T$, $\begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$;

$$\begin{aligned}x_1(t) &= c_1e^{5t} + 7c_2e^{5t} + 3c_3e^{9t}\\ x_2(t) &= 2c_1e^{5t}\\ x_3(t) &= 2c_2e^{5t} + c_3e^{9t}\end{aligned}$$

10. Characteristic equation $-\lambda^3+13\lambda^2-51\lambda+63=-(\lambda-3)^2(\lambda-7)$;

Eigenvalues $\lambda=3,3,7$;

Eigenvectors $\begin{bmatrix}5 & 2 & 0\end{bmatrix}^T$, $\begin{bmatrix}-3 & 0 & 1\end{bmatrix}^T$, $\begin{bmatrix}2 & 1 & 0\end{bmatrix}^T$;

$$\begin{aligned}x_1(t) &= 5c_1e^{5t} - 3c_2e^{3t} + 2c_3e^{7t}\\ x_2(t) &= 2c_1e^{3t} + c_3e^{7t}\\ x_3(t) &= c_2e^{3t}\end{aligned}$$

In each of Problems 11-14, the characteristic equation is $-\lambda^3-3\lambda^2-3\lambda-1=-(\lambda+1)^3$. Hence $\lambda=-1$ is a triple eigenvalue of defect 2, and we find that $(\mathbf{A}-\lambda\mathbf{I})^3=\mathbf{0}$. In each problem we start with $\mathbf{v}_3=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ and then calculate $\mathbf{v}_2=(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1=(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2\neq 0$. It follows that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1=(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2=(\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v}_3=\mathbf{0}$, so the vector $\mathbf{v}_1$ (if nonzero) is an ordinary eigenvector associated with the triple eigenvalue λ. Hence $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a length 3 chain of generalized eigenvectors, and the corresponding general solution is described by

$$\mathbf{x}(t)=e^{-t}\left[c_1\mathbf{v}_1+c_2(\mathbf{v}_1t+\mathbf{v}_2)+c_3\left(\mathbf{v}_1\frac{t^2}{2}+\mathbf{v}_2t+\mathbf{v}_3\right)\right].$$

We give the scalar components $x_1(t)$, $x_2(t)$, $x_3(t)$ of $\mathbf{x}(t)$.

11. $\mathbf{v}_1=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$, $\mathbf{v}_2=\begin{bmatrix}-2 & -1 & 1\end{bmatrix}^T$, $\mathbf{v}_3=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$;

$$x_1(t)=e^{-t}(-2c_2+c_3-2c_3t),\quad x_2(t)=e^{-t}\left(c_1-c_2+c_2t-c_3t+c_3\frac{t^2}{2}\right),\quad x_3(t)=e^{-t}(c_2+c_3t).$$

12. $\mathbf{v}_1=[1\ \ 1\ \ 0]^{\mathrm{T}}$, $\mathbf{v}_2=[0\ \ 0\ \ 1]^{\mathrm{T}}$, $\mathbf{v}_3=[1\ \ 0\ \ 0]^{\mathrm{T}}$;

$$x_1(t)=e^{-t}\left(c_1+c_3+c_2t+c_3\frac{t^2}{2}\right),\quad x_2(t)=e^{-t}\left(c_1+c_2t+c_3\frac{t^2}{2}\right),\quad x_3(t)=e^{-t}(c_2+c_3t).$$

13. Here we are stymied initially, because if $\mathbf{v}_3=[1\ \ 0\ \ 0]^{\mathrm{T}}$, then $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3=\mathbf{0}$ does not qualify as a (nonzero) generalized eigenvector. We therefore make a fresh start with $\mathbf{v}_3=[0\ \ 1\ \ 0]^{\mathrm{T}}$, and now we get the desired nonzero generalized eigenvectors upon successive multiplication by $\mathbf{A}-\lambda\mathbf{I}$.

$$\mathbf{v}_1=[1\ \ 0\ \ 0]^{\mathrm{T}},\ \mathbf{v}_2=[0\ \ 2\ \ 1]^{\mathrm{T}},\ \mathbf{v}_3=[0\ \ 1\ \ 0]^{\mathrm{T}};$$

$$x_1(t)=e^{-t}\left(c_1+c_2t+c_3\frac{t^2}{2}\right),\quad x_2(t)=e^{-t}(2c_2+c_3+\ 2c_3t),\quad x_3(t)=e^{-t}(c_2+c_3t).$$

14. $\mathbf{v}_1=[5\ \ -25\ \ -5]^{\mathrm{T}}$, $\mathbf{v}_2=[1\ \ -5\ \ 4]^{\mathrm{T}}$, $\mathbf{v}_3=[1\ \ 0\ \ 0]^{\mathrm{T}}$;

$$x_1(t)=e^{-t}\left(5c_1+c_2+c_3+\ 5c_2t+c_3t+5c_3\frac{t^2}{2}\right)$$

$$x_2(t)=e^{-t}\left(-25c_1-5c_2-25c_2t-5c_3t-25c_3\frac{t^2}{2}\right)$$

$$x_3(t)=e^{-t}\left(-5c_1+4c_2-5c_2t+4c_3t-5c_3\frac{t^2}{2}\right)$$

In each of Problems 15-18, the characteristic equation is $-\lambda^3+3\lambda^2-3\lambda+1=-(\lambda-1)^3$. Hence $\lambda=1$ is a triple eigenvalue of defect 1, and we find that $(\mathbf{A}-\lambda\mathbf{I})^2=\mathbf{0}$. First we find the two linearly independent (ordinary) eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ associated with λ. Then we start with $\mathbf{v}_2=[1\ \ 0\ \ 0]^{\mathrm{T}}$ and calculate $\mathbf{v}_1=(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2\neq\mathbf{0}$. It follows that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1=(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2=\mathbf{0}$, so $\mathbf{v}_1$ is an ordinary eigenvector associated with λ. However, $\mathbf{v}_1$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, so $\mathbf{v}_1e^t$ is a linear combination of the independent solutions $\mathbf{u}_1e^t$ and $\mathbf{u}_2e^t$. But $\{\mathbf{v}_1,\mathbf{v}_2\}$ is a length 2 chain of generalized eigenvectors associated with λ, so $(\mathbf{v}_1t+\mathbf{v}_2)e^t$ is the desired third independent solution. The corresponding general solution is described by

$$\mathbf{x}(t)=e^t\left[c_1\mathbf{u}_1+c_2\mathbf{u}_2+c_3\left(\mathbf{v}_1t+\mathbf{v}_2\right)\right].$$

We give the scalar components $x_1(t)$, $x_2(t)$, $x_3(t)$ of $\mathbf{x}(t)$.

15. $\mathbf{u}_1=[3\quad -1\quad 0]^T$, $\mathbf{u}_2=[0\quad 0\quad 1]^T$;

$\mathbf{v}_1=[-3\quad 1\quad 1]^T$, $\mathbf{v}_2=[1\quad 0\quad 0]^T$;

$$x_1(t)=e^t(3c_1+c_3-3c_3t),\quad x_2(t)=e^t(-c_1+c_3t),\quad x_3(t)=e^t(c_2+c_3t).$$

16. $\mathbf{u}_1=[3\quad -2\quad 0]^T$, $\mathbf{u}_2=[3\quad 0\quad -2]^T$;

$\mathbf{v}_1=[0\quad -2\quad 2]^T$, $\mathbf{v}_2=[1\quad 0\quad 0]^T$;

$$x_1(t)=e^t(3c_1+3c_2+c_3),\quad x_2(t)=e^t(-2c_1-2c_3t),\quad x_3(t)=e^t(-2c_2+2c_3t).$$

17. $\mathbf{u}_1=[2\quad 0\quad -9]^T$, $\mathbf{u}_2=[1\quad -3\quad 0]^T$;

$\mathbf{v}_1=[0\quad 6\quad -9]^T$; $\mathbf{v}_2=[0\quad 1\quad 0]^T$;

(Either $\mathbf{v}_2=[1\quad 0\quad 0]^T$ or $\mathbf{v}_2=[0\quad 0\quad 1]^T$ can be used also, but they yield different forms of the solution than given in the book's answer section.)

$$x_1(t)=e^t(2c_1+c_2),\quad x_2(t)=e^t(-3c_2+c_3+6c_3t),\quad x_3(t)=e^t(-9c_1-9c_3t).$$

18. $\mathbf{u}_1=[-1\quad 0\quad 1]^T$, $\mathbf{u}_2=[-2\quad 1\quad 0]^T$;

$\mathbf{v}_1=[0\quad 1\quad -2]^T$, $\mathbf{v}_2=[1\quad 0\quad 0]^T$;

$$x_1(t)=e^t(-c_1-2c_2+c_3),\quad x_2(t)=e^t(c_2+c_3t),\quad x_3(t)=e^t(c_1-2c_3t).$$

19. Characteristic equation $\lambda^4-2\lambda^2+1=0$;

Double eigenvalue $\lambda=-1$ with eigenvectors $\mathbf{v}_1=[1\quad 0\quad 0\quad 1]^T$ and $\mathbf{v}_2=[0\quad 0\quad 1\quad 0]^T$;

Double eigenvalue $\lambda=+1$ with eigenvectors $\mathbf{v}_3=[0\quad 1\quad 0\quad -2]^T$ and $\mathbf{v}_4=[1\quad 0\quad 3\quad 0]^T$;

General solution

$$\mathbf{x}(t)=e^{-t}(c_1\mathbf{v}_1+c_2\mathbf{v}_2)+e^t(c_3\mathbf{v}_3+c_4\mathbf{v}_4);$$

Scalar components

$$x_1(t)=c_1e^{-t}+c_4e^t,\quad x_2(t)=c_3e^t,\quad x_3(t)=c_2e^{-t}+3c_4e^t,\quad x_4(t)=c_1e^{-t}-2c_3e^t.$$

20. Characteristic equation $\lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 16 = (\lambda - 2)^4 = 0$;

Eigenvalue $\lambda = 2$ with multiplicity 4 and defect 3.

We find that $(\mathbf{A} - \lambda\mathbf{I})^3 \neq \mathbf{0}$ but $(\mathbf{A} - \lambda\mathbf{I})^4 = \mathbf{0}$. We therefore start with $\mathbf{v}_4 = [0 \quad 0 \quad 0 \quad 1]^T$ and define $\mathbf{v}_3 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_4$, $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$, $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$, and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$. This gives the length 4 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ with

$$\mathbf{v}_1 = [1 \quad 0 \quad 0 \quad 1]^T, \quad \mathbf{v}_2 = [0 \quad 1 \quad 0 \quad 0]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 1 \quad 0]^T, \quad \mathbf{v}_4 = [0 \quad 0 \quad 0 \quad 1]^T.$$

The corresponding general solution is given by

$$\mathbf{x}(t) = e^{-t}\left[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3\left(\mathbf{v}_1\frac{t^2}{2} + \mathbf{v}_2 t + \mathbf{v}_3\right) + c_4\left(\mathbf{v}_1\frac{t^3}{6} + \mathbf{v}_2\frac{t^2}{2} + \mathbf{v}_3 t + \mathbf{v}_4\right)\right],$$

with scalar components

$$x_1(t) = e^{2t}\left(c_1 + c_3 + c_2 t + c_4 t + c_3\frac{t^2}{2} + c_4\frac{t^3}{6}\right)$$
$$x_2(t) = e^{2t}\left(c_2 + c_3 t + c_4\frac{t^2}{2}\right)$$
$$x_3(t) = e^{2t}(c_3 + c_4 t)$$
$$x_4(t) = e^{2t}(c_4)$$

21. Characteristic equation $\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = (\lambda - 1)^4 = 0$;

Eigenvalue $\lambda = 1$ with multiplicity 4 and defect 2.

We find that $(\mathbf{A} - \lambda\mathbf{I})^2 \neq \mathbf{0}$ but $(\mathbf{A} - \lambda\mathbf{I})^3 = \mathbf{0}$. We therefore start with $\mathbf{v}_3 = [1 \quad 0 \quad 0 \quad 0]^T$ and define $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$, thereby obtaining the length 3 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [0 \quad 0 \quad 0 \quad 1]^T, \quad \mathbf{v}_2 = [-2 \quad 1 \quad 1 \quad 0]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0 \quad 0]^T.$$

Then we find the second ordinary eigenvector $\mathbf{v}_4 = [0 \quad 0 \quad 1 \quad 0]^T$. The corresponding general solution

$$\mathbf{x}(t) = e^{t}\left[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3\left(\mathbf{v}_1\frac{t^2}{2} + \mathbf{v}_2 t + \mathbf{v}_3\right) + c_4\mathbf{v}_4\right]$$

has scalar components

$$x_1(t)=e^t(-2c_2+c_3-2c_3t)$$
$$x_2(t)=e^t(c_2+c_3t)$$
$$x_3(t)=e^t(c_2+c_4+c_3t)$$
$$x_4(t)=e^t\left(c_1+c_2t+c_3\frac{t^2}{2}\right)$$

22. Same eigenvalue and chain structure as in Problem 21, but with generalized eigenvectors $\mathbf{v}_1=[1\ \ 0\ \ 0\ \ -2]^T$, $\mathbf{v}_2=[3\ \ -2\ \ 1\ \ -6]^T$, $\mathbf{v}_3=[0\ \ 1\ \ 0\ \ 0]^T$, $\mathbf{v}_4=[1\ \ 0\ \ 0\ \ 0]^T$, where $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a length 3 chain and $\mathbf{v}_4$ is an ordinary eigenvector. The general solution $\mathbf{x}(t)$ defined as in Problem 21 has scalar components

$$x_1(t)=e^t\left(c_1+\ 3c_2+c_4+c_2t+\ 3c_3t+c_3\frac{t^2}{2}\right)$$
$$x_2(t)=e^t(-2c_2+c_3-2c_3t)$$
$$x_3(t)=e^t(c_2+c_3t)$$
$$x_4(t)=e^t(-2c_1-6c_2-2c_2t-6c_3t-c_3t^2)$$

In Problems 23 and 24 there are only two distinct eigenvalues λ_1 and λ_2. However, the eigenvector equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ yields the three linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ that are given. We list the scalar components of the corresponding general solution $\mathbf{x}(t)=c_1\mathbf{v}_1e^{\lambda_1 t}+c_2\mathbf{v}_2e^{\lambda_2 t}+c_3\mathbf{v}_3e^{\lambda_2 t}$.

23. $\lambda_1=-1$: $\{\mathbf{v}_1\}$ with $\mathbf{v}_1=[1\ \ -1\ \ 2]^T$;

$\lambda_2=3$: $\{\mathbf{v}_2\}$ with $\mathbf{v}_2=[4\ \ 0\ \ 9]^T$ and $\{\mathbf{v}_3\}$ with $\mathbf{v}_3=[0\ \ 2\ \ 1]^T$.

Scalar components:

$$\begin{aligned} x_1(t) &= c_1e^{-t} + 4c_2e^{3t} \\ x_2(t) &= -c_1e^{-t} + 2c_3e^{3t} \\ x_3(t) &= 2c_1e^{-t} + 9c_2e^{3t} + c_3e^{3t} \end{aligned}$$

24. $\lambda_1=-2$: $\{\mathbf{v}_1\}$ with $\mathbf{v}_1=[5\ \ 3\ \ -3]^T$;

$\lambda_2=3$: $\{\mathbf{v}_2\}$ with $\mathbf{v}_2=[4\ \ 0\ \ -1]^T$ and $\{\mathbf{v}_3\}$ with $\mathbf{v}_3=[2\ \ -1\ \ 0]^T$.

Scalar components:

$$\begin{aligned}
x_1(t) &= 5c_1e^{-2t} + 4c_2e^{3t} + 2c_3e^{3t} \\
x_2(t) &= 3c_1e^{-2t} \qquad\qquad - c_3e^{3t} \\
x_3(t) &= -3c_1e^{-2t} - c_2e^{3t}
\end{aligned}$$

In Problems 25, 26, and 28 there is given a single eigenvalue λ of multiplicity 3. We find that $(\mathbf{A}-\lambda\mathbf{I})^2 \neq \mathbf{0}$ but $(\mathbf{A}-\lambda\mathbf{I})^3 = \mathbf{0}$. We therefore start with $\mathbf{v}_3 = [1 \quad 0 \quad 0]^{\mathrm{T}}$ and define $\mathbf{v}_2 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$, thereby obtaining the length 3 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of generalized eigenvectors based on the ordinary eigenvector $\mathbf{v}_1$. We list the scalar components of the corresponding general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda t} + c_2(\mathbf{v}_1t + \mathbf{v}_2)e^{\lambda t} + c_3\left(\mathbf{v}_1\frac{t^2}{2} + \mathbf{v}_2t + \mathbf{v}_3\right)e^{\lambda t}.$$

25. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [-1 \quad 0 \quad -1]^{\mathrm{T}}, \quad \mathbf{v}_2 = [-4 \quad -1 \quad 0]^{\mathrm{T}}, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^{\mathrm{T}}$$

Scalar components:

$$\begin{aligned}
x_1(t) &= e^{2t}\left(-c_1 - 4c_2 + c_3 - c_2t - 4c_3t - c_3\frac{t^2}{2}\right) \\
x_2(t) &= e^{2t}(-c_2 - c_3t) \\
x_3(t) &= e^{2t}\left(-c_1 - c_2t - c_3\frac{t^2}{2}\right)
\end{aligned}$$

26. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [0 \quad 2 \quad 2]^{\mathrm{T}}, \quad \mathbf{v}_2 = [2 \quad 1 \quad -3]^{\mathrm{T}}, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^{\mathrm{T}};$$

General solution:

$$\begin{aligned}
x_1(t) &= e^{3t}(2c_2 + c_3 + 2c_3t) \\
x_2(t) &= e^{3t}(2c_1 + c_2 + 2c_2t + c_3t + c_3t^2) \\
x_3(t) &= e^{3t}(2c_1 - 3c_2 + 2c_2t - 3c_3t + c_3t^2)
\end{aligned}$$

27. We find that the triple eigenvalue $\lambda = 2$ has the two linearly independent eigenvectors $[1 \quad 1 \quad 0]^{\mathrm{T}}$ and $[-1 \quad 0 \quad 1]^{\mathrm{T}}$. Next we find that $(\mathbf{A}-\lambda\mathbf{I}) \neq \mathbf{0}$ but $(\mathbf{A}-\lambda\mathbf{I})^2 = \mathbf{0}$. We therefore start with $\mathbf{v}_2 = [1 \quad 0 \quad 0]^{\mathrm{T}}$ and define

$$\mathbf{v}_1 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 = [5 \quad -3 \quad 8]^{\mathrm{T}} \neq \mathbf{0},$$

thereby obtaining the length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ of generalized eigenvectors based on the ordinary eigenvector $\mathbf{v}_1$. If we take $\mathbf{v}_3 = [1 \quad 1 \quad 0]^T$, then the general solution

$$\mathbf{x}(t) = e^{2t}\left[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3\mathbf{v}_3\right]$$

has scalar components

$$\begin{aligned} x_1(t) &= e^{2t}(-5c_1 + c_2 + c_3 - 5c_2 t) \\ x_2(t) &= e^{2t}(3c_1 + 3c_2 t) \\ x_3(t) &= e^{2t}(8c_1 + 8c_2 t) \end{aligned}$$

28. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [119 \quad -289 \quad 0]^T, \quad \mathbf{v}_2 = [-17 \quad 34 \quad 17]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T;$$

General solution

$$\begin{aligned} x_1(t) &= e^{2t}\left(119c_1 - 17c_2 + c_3 + 119c_2 t - 17c_3 t + 119c_3\frac{t^2}{2}\right) \\ x_2(t) &= e^{2t}\left(-289c_1 + 34c_2 - 289c_2 t + 34c_3 t - 289c_3\frac{t^2}{2}\right) \\ x_3(t) &= e^{2t}(17c_2 + 17c_3 t) \end{aligned}$$

In Problems 29 and 30 the matrix **A** has two distinct eigenvalues λ_1 and λ_2 each having multiplicity 2 and defect 1. First, we select $\mathbf{v}_2$ so that $\mathbf{v}_1 = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$ but $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a length 2 chain based on $\mathbf{v}_1$. Next, we select $\mathbf{u}_2$ so that $\mathbf{u}_1 = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u}_2 \neq \mathbf{0}$ but $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u}_1 = \mathbf{0}$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on $\mathbf{u}_1$. We give the scalar components of the corresponding general solution

$$\mathbf{x}(t) = e^{\lambda_1 t}\left[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2)\right] + e^{\lambda_2 t}\left[c_3\mathbf{u}_1 + c_4(\mathbf{u}_1 t + \mathbf{u}_2)\right].$$

29. $\lambda = -1$: $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = [1 \quad -3 \quad -1 \quad -2]^T$ and $\mathbf{v}_2 = [0 \quad 1 \quad 0 \quad 0]^T$;

$\lambda = 2$: $\{\mathbf{u}_1, \mathbf{u}_2\}$ with $\mathbf{u}_1 = [0 \quad -1 \quad 1 \quad 0]^T$ and $\mathbf{u}_2 = [0 \quad 0 \quad 2 \quad 1]^T$;

Scalar components

$$\begin{aligned} x_1(t) &= e^{-t}(c_1 + c_2 t) \\ x_2(t) &= e^{-t}(-3c_1 + c_2 - 3c_2 t) + e^{2t}(-c_3 - c_4 t) \\ x_3(t) &= e^{-t}(-c_1 - c_2 t) + e^{2t}(c_3 + 2c_4 + c_4 t) \\ x_4(t) &= e^{-t}(-2c_1 - 2c_2 t) + e^{2t}(c_4) \end{aligned}$$

30. $\lambda = -1$: $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = [0 \quad 1 \quad -1 \quad -3]^T$ and $\mathbf{v}_2 = [0 \quad 0 \quad 1 \quad 2]^T$;

$\lambda = 2$: $\{\mathbf{u}_1, \mathbf{u}_2\}$ with $\mathbf{u}_1 = [-1 \quad 0 \quad 0 \quad 0]^T$ and $\mathbf{u}_2 = [0 \quad 0 \quad 3 \quad 5]^T$;

Scalar components

$$\begin{aligned}
x_1(t) &= e^{2t}(-c_3 - c_4 t) \\
x_2(t) &= e^{-t}(c_1 + c_2 t) \\
x_3(t) &= e^{-t}(-c_1 + c_2 - c_2 t) + e^{2t}(3c_4) \\
x_4(t) &= e^{-t}(-3c_1 + 2c_2 - 3c_2 t) + e^{2t}(5c_4)
\end{aligned}$$

31. We have the single eigenvalue $\lambda = 1$ of multiplicity 4. Starting with $\mathbf{v}_3 = [1 \quad 0 \quad 0 \quad 0]^T$, we calculate $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$, and find that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}$. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain based on the ordinary eigenvector $\mathbf{v}_1$. Next, the eigenvector equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ yields the second linearly independent eigenvector $\mathbf{v}_4 = [0 \quad 1 \quad 3 \quad 0]^T$. With

$$\begin{aligned}
\mathbf{v}_1 &= [42 \quad 7 \quad -21 \quad -42]^T, & \mathbf{v}_2 &= [34 \quad 22 \quad -10 \quad -27]^T, \\
\mathbf{v}_3 &= [1 \quad 0 \quad 0 \quad 0]^T, & \mathbf{v}_4 &= [0 \quad 1 \quad 3 \quad 0]^T
\end{aligned}$$

the general solution

$$\mathbf{x}(t) = e^t \left[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3\left(\mathbf{v}_1 \frac{t^2}{2} + \mathbf{v}_2 t + \mathbf{v}_3\right) + c_4\mathbf{v}_4 \right]$$

has scalar components

$$\begin{aligned}
x_1(t) &= e^t\left(42c_1 + 34c_2 + c_3 + 42c_2 t + 34c_3 t + 21c_3 t^2\right) \\
x_2(t) &= e^t\left(7c_1 + 22c_2 + c_4 + 7c_2 t + 22c_3 t + 7c_3 \frac{t^2}{2}\right) \\
x_3(t) &= e^t\left(-21c_1 - 10c_2 + 3c_4 - 21c_2 t - 10c_3 t - 21c_3 \frac{t^2}{2}\right) \\
x_4(t) &= e^t\left(-42c_1 - 27c_2 - 42c_2 t - 27c_3 t - 21c_3 t^2\right)
\end{aligned}$$

32. Here we find that the matrix **A** has five linearly independent eigenvectors:

$\lambda = 2$: Eigenvectors $\mathbf{v}_1 = [8 \quad 0 \quad -3 \quad 1 \quad 0]^T$ and $\mathbf{v}_2 = [1 \quad 0 \quad 0 \quad 0 \quad 3]^T$;

$\lambda = 3$: Eigenvectors $\mathbf{v}_3 = [3 \quad -2 \quad -1 \quad 0 \quad 0]^T$, $\mathbf{v}_4 = [2 \quad -2 \quad 0 \quad -3 \quad 0]^T$, and $\mathbf{v}_5 = [1 \quad -1 \quad 0 \quad 0 \quad 3]^T$.

The general solution

$$\mathbf{x}(t)=e^{2t}\left(c_1\mathbf{v}_1+c_2\mathbf{v}_2\right)\ +e^{3t}\left(c_3\mathbf{v}_3+c_4\mathbf{v}_4+c_5\mathbf{v}_5\right)$$

has scalar components

$$\begin{aligned}
x_1(t) &= e^{2t}\left(8c_1+c_2\right)+e^{3t}\left(3c_3+2c_4+c_5\right)\\
x_2(t) &= e^{3t}\left(-2c_3-2c_4-c_5\right)\\
x_3(t) &= e^{2t}\left(-3c_1\right)+e^{3t}\left(-c_3\right)\\
x_4(t) &= e^{2t}\left(c_1\right)+e^{3t}\left(-3c_4\right)\\
x_5(t) &= e^{2t}\left(3c_2\right)+e^{3t}\left(3c_5\right)
\end{aligned}$$

33. The chain $\{\mathbf{v}_1,\mathbf{v}_2\}$ was found using the matrices

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}4i & -4 & 1 & 0\\ 4 & 4i & 0 & 1\\ 0 & 0 & 4i & -4\\ 0 & 0 & 4 & 4i\end{bmatrix}\rightarrow\begin{bmatrix}1 & i & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

and

$$(\mathbf{A}-\lambda\mathbf{I})^2 = \begin{bmatrix}-32 & -32i & 8i & -8\\ 32i & -32 & 8 & 8i\\ 0 & 0 & -32 & -32i\\ 0 & 0 & 32i & -32\end{bmatrix}\rightarrow\begin{bmatrix}1 & i & 0 & 0\\ 0 & 0 & 1 & i\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

where $\rightarrow$ signifies reduction to row-echelon form. The resulting real-valued solution vectors are

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{3t}\left[\begin{matrix}\cos 4t & \sin 4t & 0 & 0\end{matrix}\right]^{\mathrm{T}}\\
\mathbf{x}_2(t) &= e^{3t}\left[\begin{matrix}-\sin 4t & \cos 4t & 0 & 0\end{matrix}\right]^{\mathrm{T}}\\
\mathbf{x}_3(t) &= e^{3t}\left[\begin{matrix}t\cos 4t & t\sin 4t & \cos 4t & \sin 4t\end{matrix}\right]^{\mathrm{T}}\\
\mathbf{x}_4(t) &= e^{3t}\left[\begin{matrix}-t\sin 4t & t\cos 4t & -\sin 4t & \cos 4t\end{matrix}\right]^{\mathrm{T}}
\end{aligned}$$

34. The chain $\{\mathbf{v}_1,\mathbf{v}_2\}$ was found using the matrices

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}3i & 0 & -8 & -3\\ -18 & -3-3i & 0 & 0\\ -9 & -3 & -27-3i & -9\\ 33 & 10 & 90 & 30-3i\end{bmatrix}\rightarrow\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 3+3i\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

and

$$(\mathbf{A}-\lambda\mathbf{I})^2 = \begin{bmatrix} -36 & -6 & -54+48i & -18+18i \\ 54+108i & 18i & 144 & 54 \\ 54i & 18i & -18+162i & 54i \\ -198i & -60i & 6-540i & -18-180i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3i & -i \\ 0 & 1 & 9+10i & 3+3i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\to$ signifies reduction to row-echelon form. The resulting real-valued solution vectors are

$$\mathbf{x}_1(t) = e^{2t}\left[\begin{matrix} \sin 3t & 3\cos 3t - 3\sin 3t & 0 & \sin 3t \end{matrix}\right]^{\mathrm{T}}$$
$$\mathbf{x}_2(t) = e^{2t}\left[\begin{matrix} -\cos 3t & 3\sin 3t + 3\cos 3t & 0 & -\cos 3t \end{matrix}\right]^{\mathrm{T}}$$
$$\mathbf{x}_3(t) = e^{2t}\left[\begin{matrix} 3\cos 3t + t\sin 3t & (3t-10)\cos 3t - (3t+9)\sin 3t & \sin 3t & t\sin 3t \end{matrix}\right]^{\mathrm{T}}$$
$$\mathbf{x}_4(t) = e^{2t}\left[\begin{matrix} -t\cos 3t + 3\sin 3t & (3t+9)\cos 3t + (3t-10)\sin 3t & -\cos 3t & -t\cos 3t \end{matrix}\right]^{\mathrm{T}}$$

35. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix}$$

has the following eigenvalues and corresponding eigenvectors:

λ	Corresponding eigenvector(s)
0	$\mathbf{v}_1 = [1 \quad 1 \quad 0 \quad 0]^{\mathrm{T}}$
−1	$\mathbf{v}_2 = [1 \quad 0 \quad -1 \quad 0]^{\mathrm{T}}$ and $\mathbf{v}_3 = [0 \quad 1 \quad 0 \quad -1]^{\mathrm{T}}$
−2	$\mathbf{v}_4 = [1 \quad -1 \quad -2 \quad 2]^{\mathrm{T}}$

When we impose the given initial conditions on the general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 e^{-t} + c_3\mathbf{v}_3 e^{-t} + c_4\mathbf{v}_4 e^{-2t}$$

we find that $c_1 = v_0$, $c_2 = c_3 = -v_0$, $c_4 = 0$. Hence the position functions of the two masses are given by

$$x_1(t) = x_2(t) = v_0\left(1 - e^{-t}\right).$$

Each mass travels a distance v_0 before stopping.

36. The coefficient matrix is the same as in Problem 35 except that $a_{44} = -1$. Now the matrix **A** has the eigenvalue $\lambda = 0$ with eigenvector $\mathbf{v}_0 = [1 \quad 1 \quad 0 \quad 0]^{\mathrm{T}}$, and the triple eigenvalue $\lambda = -1$ with associated length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ consisting of the generalized eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}^T, \quad \mathbf{v}_2 = \begin{bmatrix} 1 & 0 & -1 & 1 \end{bmatrix}^T, \quad \mathbf{v}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

When we impose the given initial conditions on the general solution

$$\mathbf{x}(t) = c_0\mathbf{v}_0 + e^{-t}\left[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3\left(\mathbf{v}_1\frac{t^2}{2} + \mathbf{v}_2 t + \mathbf{v}_3\right)\right]$$

we find that $c_0 = 2v_0$, $c_1 = -2v_0$, and $c_2 = c_3 = -v_0$. Hence the position functions of the two masses are given by

$$x_1(t) = v_0\left(2 - 2e^{-t} - te^{-t}\right), \quad x_2(t) = v_0\left(2 - 2e^{-t} - te^{-t} - \frac{t^2}{2}e^{-t}\right)$$

Each travels a distance $2v_0$ before stopping.

SECTION 5.6

MATRIX EXPONENTIALS AND LINEAR SYSTEMS

In Problems 1–8 we first use the eigenvalues and eigenvectors of the coefficient matrix **A** to find first a fundamental matrix $\Phi(t)$ for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then we apply the formula $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$ to find the solution vector $\mathbf{x}(t)$ that satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Formulas (11) and (12) in the text provide inverses of 2×2 and 3×3 matrices.

1. Eigensystem: $\lambda_1 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$; $\lambda_2 = 3$, $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5e^t + e^{3t} \\ -5e^t + e^{3t} \end{bmatrix}$$

2. Eigensystem: $\lambda_1 = 0$, $\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$; $\lambda_2 = 4$, $\lambda_2 = 4$, $\mathbf{v}_2 = \begin{bmatrix} 1 & -2 \end{bmatrix}^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 + 5e^{4t} \\ 6 - 10e^{4t} \end{bmatrix}$$

3. Eigensystem: $\lambda = 4i$, $\mathbf{v} = \begin{bmatrix}1+2i & 2\end{bmatrix}^{\mathrm{T}}$;

$$\Phi(t) = \left[\operatorname{Re}\left(\mathbf{v}e^{\lambda t}\right) \quad \operatorname{Im}\left(\mathbf{v}e^{\lambda t}\right)\right] = \begin{bmatrix}\cos 4t - 2\sin 4t & 2\cos 4t + \sin 4t \\ 2\cos 4t & 2\sin 4t\end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix}\cos 4t - 2\sin 4t & 2\cos 4t + \sin 4t \\ 2\cos 4t & 2\sin 4t\end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix}0 & 2 \\ 2 & -1\end{bmatrix} \cdot \begin{bmatrix}0 \\ 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-5\sin 4t \\ 4\cos 4t - 2\sin 4t\end{bmatrix}$$

4. Eigensystem: $\lambda = 2, 2$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = \begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}}$, $\mathbf{v}_2 = \begin{bmatrix}1 & 0\end{bmatrix}^{\mathrm{T}}$;

$$\Phi(t) = \left[\mathbf{v}_1 e^{\lambda t} \quad (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t}\right] = e^{2t}\begin{bmatrix}1 & 1+t \\ 1 & t\end{bmatrix}$$

$$\mathbf{x}(t) = e^{2t}\begin{bmatrix}1 & 1+t \\ 1 & t\end{bmatrix} \cdot \begin{bmatrix}0 & 1 \\ 1 & -1\end{bmatrix} \cdot \begin{bmatrix}1 \\ 0\end{bmatrix} = e^{2t}\begin{bmatrix}1+t \\ t\end{bmatrix}$$

5. Eigensystem: $\lambda = 3i$, $\mathbf{v} = \begin{bmatrix}-1+i & 3\end{bmatrix}^{\mathrm{T}}$;

$$\Phi(t) = \left[\operatorname{Re}\left(\mathbf{v}e^{\lambda t}\right) \quad \operatorname{Im}\left(\mathbf{v}e^{\lambda t}\right)\right] = \begin{bmatrix}-\cos 3t - \sin 3t & \cos 3t - \sin 3t \\ 3\cos 3t & 3\sin 3t\end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix}-\cos 3t - \sin 3t & \cos 3t - \sin 3t \\ 3\cos 3t & 3\sin 3t\end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix}0 & 1 \\ 3 & 1\end{bmatrix} \cdot \begin{bmatrix}1 \\ -1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}3\cos 3t - \sin 3t \\ -3\cos 3t + 6\sin 3t\end{bmatrix}$$

6. Eigensystem: $\lambda = 5 + 4i$, $\mathbf{v} = \begin{bmatrix}1+2i & 2\end{bmatrix}^{\mathrm{T}}$;

$$\Phi(t) = \left[\operatorname{Re}\left(\mathbf{v}e^{\lambda t}\right) \quad \operatorname{Im}\left(\mathbf{v}e^{\lambda t}\right)\right] = e^{5t}\begin{bmatrix}\cos 4t - 2\sin 4t & 2\cos 4t + 2\sin 4t \\ 2\cos 4t & 2\sin 4t\end{bmatrix}$$

$$\mathbf{x}(t) = e^{5t}\begin{bmatrix}\cos 4t - 2\sin 4t & 2\cos 4t + 2\sin 4t \\ 2\cos 4t & 2\sin 4t\end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix}0 & 2 \\ 2 & -4\end{bmatrix} \cdot \begin{bmatrix}2 \\ 0\end{bmatrix} = 2e^{5t}\begin{bmatrix}\cos 4t + \sin 4t \\ \sin 4t\end{bmatrix}$$

7. Eigensystem:

$\lambda_1 = 0, \quad \mathbf{v}_1 = \begin{bmatrix}6 & 2 & 5\end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 1, \quad \mathbf{v}_2 = \begin{bmatrix}3 & 1 & 2\end{bmatrix}^{\mathrm{T}}; \quad \lambda_3 = -1, \quad \mathbf{v}_3 = \begin{bmatrix}2 & 1 & 2\end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \left[e^{\lambda_1 t}\mathbf{v}_1 \quad e^{\lambda_2 t}\mathbf{v}_2 \quad e^{\lambda_3 t}\mathbf{v}_3\right] = \begin{bmatrix}6 & 3e^{t} & 2e^{-t} \\ 2 & e^{t} & e^{-t} \\ 5 & 2e^{t} & 2e^{-t}\end{bmatrix}$$

$$\mathbf{x}(t)=\begin{bmatrix}6 & 3e^{t} & 2e^{-t}\\ 2 & e^{t} & e^{-t}\\ 5 & 2e^{t} & 2e^{-t}\end{bmatrix}\cdot\begin{bmatrix}0 & -2 & 1\\ 1 & 2 & -2\\ -1 & 3 & 0\end{bmatrix}\cdot\begin{bmatrix}2\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}-12+12e^{t}+2e^{-t}\\ -4+4e^{t}+e^{-t}\\ -10+8e^{t}+2e^{-t}\end{bmatrix}$$

8. Eigensystem:

$\lambda_1=-2,\quad \mathbf{v}_1=\begin{bmatrix}0 & 1 & -1\end{bmatrix}^{\mathrm{T}};\quad \lambda_2=1,\quad \mathbf{v}_2=\begin{bmatrix}1 & -1 & 0\end{bmatrix}^{\mathrm{T}};\quad \lambda_3=3,\quad \mathbf{v}_3=\begin{bmatrix}1 & -1 & 1\end{bmatrix}^{\mathrm{T}};$

$$\Phi(t)=\begin{bmatrix}e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 & e^{\lambda_3 t}\mathbf{v}_3\end{bmatrix}=\begin{bmatrix}0 & e^{t} & e^{3t}\\ e^{-2t} & -e^{t} & -e^{3t}\\ -e^{-2t} & 0 & e^{3t}\end{bmatrix}$$

$$\mathbf{x}(t)=\begin{bmatrix}0 & e^{t} & e^{3t}\\ e^{-2t} & -e^{t} & -e^{3t}\\ -e^{-2t} & 0 & e^{3t}\end{bmatrix}\cdot\begin{bmatrix}1 & 1 & 0\\ 0 & -1 & -1\\ 1 & 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}=\begin{bmatrix}e^{t}\\ -e^{t}+e^{-2t}\\ -e^{-2t}\end{bmatrix}$$

In each of Problems 9-20 we first solve the given linear system to find two linearly independent solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, then set up the fundamental matrix $\mathbf{\Phi}(t)=\begin{bmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t)\end{bmatrix}$, and finally calculate the matrix exponential $e^{\mathbf{A}t}=\Phi(t)\Phi(0)^{-1}$.

9. Eigensystem: $\lambda_1=1,\quad \mathbf{v}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}};\quad \lambda_2=3,\quad \mathbf{v}_2=\begin{bmatrix}2 & 1\end{bmatrix}^{\mathrm{T}};$

$$\Phi(t)=\begin{bmatrix}e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}e^{t} & 2e^{3t}\\ e^{t} & e^{3t}\end{bmatrix}$$

$$e^{\mathbf{A}t}=\begin{bmatrix}e^{t} & 2e^{3t}\\ e^{t} & e^{3t}\end{bmatrix}\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}=\begin{bmatrix}-e^{t}+2e^{3t} & 2e^{t}-2e^{3t}\\ -e^{t}+e^{3t} & 2e^{t}-e^{3t}\end{bmatrix}$$

10. Eigensystem: $\lambda_1=0,\quad \mathbf{v}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}};\quad \lambda_2=2,\quad \mathbf{v}_2=\begin{bmatrix}3 & 2\end{bmatrix}^{\mathrm{T}};$

$$\Phi(t)=\begin{bmatrix}e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}1 & 3e^{2t}\\ 1 & 2e^{2t}\end{bmatrix}$$

$$e^{\mathbf{A}t}=\begin{bmatrix}1 & 3e^{2t}\\ 1 & 2e^{2t}\end{bmatrix}\begin{bmatrix}-2 & 3\\ 1 & -1\end{bmatrix}=\begin{bmatrix}-2+3e^{2t} & 3-3e^{2t}\\ -2+2e^{2t} & 3-2e^{2t}\end{bmatrix}$$

11. Eigensystem: $\lambda_1=2,\quad \mathbf{v}_1=\begin{bmatrix}1 & 1\end{bmatrix}^{\mathrm{T}};\quad \lambda_2=3,\quad \mathbf{v}_2=\begin{bmatrix}3 & 2\end{bmatrix}^{\mathrm{T}};$

$$\Phi(t)=\begin{bmatrix}e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}e^{2t} & 3e^{3t}\\ e^{2t} & 2e^{3t}\end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{bmatrix}$$

12. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 2, \quad \mathbf{v}_2 = \begin{bmatrix} 4 & 3 \end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{t} & 4e^{2t} \\ e^{t} & 3e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{t} & 4e^{2t} \\ e^{t} & 3e^{2t} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3e^{t} + 4e^{2t} & 4e^{t} - 4e^{2t} \\ -3e^{t} + 3e^{2t} & 4e^{t} - 3e^{2t} \end{bmatrix}$$

13. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 3, \quad \mathbf{v}_2 = \begin{bmatrix} 4 & 3 \end{bmatrix}^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{t} & 4e^{3t} \\ e^{t} & 3e^{3t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{t} & 4e^{3t} \\ e^{t} & 3e^{3t} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3e^{t} + 4e^{3t} & 4e^{t} - 4e^{3t} \\ -3e^{t} + 3e^{3t} & 4e^{t} - 3e^{3t} \end{bmatrix}$$

14. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 2 & 3 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 3, \quad \mathbf{v}_2 = \begin{bmatrix} 3 & 4 \end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2e^{t} & 3e^{2t} \\ 3e^{t} & 4e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{t} & 3e^{2t} \\ 3e^{t} & 4e^{2t} \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -8e^{t} + 9e^{2t} & 6e^{t} - 6e^{2t} \\ -12e^{t} + 12e^{2t} & 9e^{t} - 8e^{2t} \end{bmatrix}$$

15. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 2, \quad \mathbf{v}_2 = \begin{bmatrix} 5 & 2 \end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2e^{t} & 5e^{2t} \\ e^{t} & 2e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{t} & 5e^{2t} \\ e^{t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -4e^{t} + 5e^{2t} & 10e^{t} - 10e^{2t} \\ -2e^{t} + 2e^{2t} & 5e^{t} - 4e^{2t} \end{bmatrix}$$

16. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 3 & 2 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 2, \quad \mathbf{v}_2 = \begin{bmatrix} 5 & 3 \end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 3e^{t} & 5e^{2t} \\ 2e^{t} & 3e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^t & 5e^{2t} \\ 2e^t & 3e^{2t} \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -9e^t + 10e^{2t} & 15e^t - 15e^{2t} \\ -6e^t + 6e^{2t} & 10e^t - 9e^{2t} \end{bmatrix}$$

17. Eigensystem: $\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 4, \quad \mathbf{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t} + e^{4t} & -e^{2t} + e^{4t} \\ -e^{2t} + e^{4t} & e^{2t} + e^{4t} \end{bmatrix}$$

18. Eigensystem: $\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 6, \quad \mathbf{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t} + e^{6t} & -e^{2t} + e^{6t} \\ -e^{2t} + e^{6t} & e^{2t} + e^{6t} \end{bmatrix}$$

19. Eigensystem: $\lambda_1 = 5, \quad \mathbf{v}_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 10, \quad \mathbf{v}_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{5t} & 2e^{10t} \\ -2e^{5t} & e^{10t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 2e^{10t} \\ -2e^{5t} & e^{10t} \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} e^{5t} + 4e^{10t} & -2e^{5t} + 2e^{10t} \\ -2e^{5t} + 2e^{10t} & 4e^{5t} + e^{10t} \end{bmatrix}$$

20. Eigensystem: $\lambda_1 = 5, \quad \mathbf{v}_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^{\mathrm{T}}; \quad \lambda_2 = 15, \quad \mathbf{v}_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}^{\mathrm{T}};$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{5t} & 2e^{15t} \\ -2e^{5t} & e^{15t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 2e^{15t} \\ -2e^{5t} & e^{15t} \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} e^{5t} + 4e^{15t} & -2e^{5t} + 2e^{15t} \\ -2e^{5t} + 2e^{15t} & 4e^{5t} + e^{15t} \end{bmatrix}$$

21. $\mathbf{A}^2 = \mathbf{0}$, so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}.$

22. $\mathbf{A}^2 = \mathbf{0}$, so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1+6t & 4t \\ -9t & 1-6t \end{bmatrix}.$

23. $\mathbf{A}^3 = \mathbf{0}$, so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 = \begin{bmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{bmatrix}$.

24. $\mathbf{A}^3 = \mathbf{0}$, so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 = \begin{bmatrix} 1+3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{bmatrix}$.

25. $\mathbf{A} = 2\mathbf{I} + \mathbf{B}$, where $\mathbf{B}^2 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{2\mathbf{I}t}e^{\mathbf{B}t} = \left(e^{2t}\mathbf{I}\right)\left(\mathbf{I} + \mathbf{B}t\right)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = e^{2t}\begin{bmatrix} 4+35t \\ 7 \end{bmatrix}.$$

26. $\mathbf{A} = 7\mathbf{I} + \mathbf{B}$, where $\mathbf{B}^2 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{7\mathbf{I}t}e^{\mathbf{B}t} = \left(e^{7t}\mathbf{I}\right)\left(\mathbf{I} + \mathbf{B}t\right)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{7t} & 0 \\ 11te^{7t} & e^{7t} \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 5 \\ -10 \end{bmatrix} = e^{7t}\begin{bmatrix} 5 \\ -10+55t \end{bmatrix}.$$

27. $\mathbf{A} = \mathbf{I} + \mathbf{B}$, where $\mathbf{B}^3 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{\mathbf{I}t}e^{\mathbf{B}t} = \left(e^{t}\mathbf{I}\right)\left(\mathbf{I} + t + \frac{1}{2}\mathbf{B}^2t^2\right)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{t} & 2te^{t} & \left(3t+2t^2\right)e^{t} \\ 0 & e^{t} & 2te^{t} \\ 0 & 0 & e^{t} \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = e^{t}\begin{bmatrix} 4+28t+12t^2 \\ 5+12t \\ 6 \end{bmatrix}.$$

28. $\mathbf{A} = 5\mathbf{I} + \mathbf{B}$, where $\mathbf{B}^3 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{5\mathbf{I}t}e^{\mathbf{B}t} = \left(e^{5t}\mathbf{I}\right)\left(\mathbf{I} + \mathbf{B}t + \frac{1}{2}\mathbf{B}^2t^2\right)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 0 & 0 \\ 10te^{5t} & e^{5t} & 0 \\ \left(20t+150t^2\right)e^{5t} & 30te^{5t} & e^{5t} \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix} = e^{5t}\begin{bmatrix} 40 \\ 50+400t \\ 60+2300t+6000t^2 \end{bmatrix}.$$

29. $\mathbf{A} = \mathbf{I} + \mathbf{B}$, where $\mathbf{B}^4 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{\mathbf{I}t}e^{\mathbf{B}t} = \left(e^{t}\mathbf{I}\right)\left(\mathbf{I} + \mathbf{B}t + \frac{1}{2}\mathbf{B}^2t^2 + \frac{1}{6}\mathbf{B}^3t^3\right)$. Hence

$$e^{\mathbf{A}t} = e^t \begin{bmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 1+9t+12t^2+4t^3 \\ 1+9t+6t^2 \\ 1+2t \\ 1 \end{bmatrix}.$$

30. $\mathbf{A} = 3\mathbf{I} + \mathbf{B}$, where $\mathbf{B}^4 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{3\mathbf{I}t}e^{\mathbf{B}t} = \left(e^{3t}\mathbf{I}\right)\left(\mathbf{I} + \mathbf{B}t + \frac{1}{2}\mathbf{B}^2t^2 + \frac{1}{6}\mathbf{B}^3t^3\right)$. Hence

$$e^{\mathbf{A}t} = e^{3t} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6t & 1 & 0 & 0 \\ 9t+18t^2 & 6t & 1 & 0 \\ 12t+54t^2+36t^3 & 9t+18t^2 & 6t & 1 \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e^{3t} \begin{bmatrix} 1 \\ 1+6t \\ 1+15t+18t^2 \\ 1+27t+72t^2+36t^3 \end{bmatrix}$$

33. $e^{\mathbf{A}t} = \mathbf{I}\cosh t + \mathbf{A}\sinh t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$, so the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c} = \begin{bmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{bmatrix}.$$

34. Direct calculation gives $\mathbf{A}^2 = -4\mathbf{I}$, and it follows that $\mathbf{A}^3 = -4\mathbf{A}$ and $\mathbf{A}^4 = 16\mathbf{I}$. Therefore

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t - \frac{4\mathbf{I}t^2}{2!} - \frac{4\mathbf{A}t^3}{3!} + \frac{16\mathbf{I}t^4}{4!} + \frac{16\mathbf{A}t^5}{5!} + \cdots \\ &= \mathbf{I}\left[1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} + \cdots\right] + \frac{1}{2}\mathbf{A}\left[(2t) - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} + \cdots\right] \\ &= \mathbf{I}\cos 2t + \frac{1}{2}\mathbf{A}\sin 2t. \end{aligned}$$

In Problems 35–40 we give first the linearly independent generalized eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ of the matrix $\mathbf{A}$ and the corresponding solution vectors $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ defined by Eq. (34) in the text, then the fundamental matrix $\Phi(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{bmatrix}$. Finally we calculate the exponential matrix $e^{\mathbf{A}t} = \Phi(t)\Phi(0)^{-1}$.

35. $\lambda = 3$: $\mathbf{u}_1 = \begin{bmatrix} 4 & 0 \end{bmatrix}^T$, $\mathbf{u}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$;

$\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\left[\mathbf{u}_2 + (\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_2 t\right]$$

$$\Phi(t) = \left[\mathbf{x}_1(t) \quad \mathbf{x}_2(t)\right] = e^{3t}\begin{bmatrix} 4 & 4t \\ 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{3t}\begin{bmatrix} 4 & 4t \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 & 4t \\ 0 & 1 \end{bmatrix}$$

36. $\lambda = 1$: $\mathbf{u}_1 = [8 \quad 0 \quad 0]^{\mathrm{T}}$, $\mathbf{u}_2 = [5 \quad 4 \quad 0]^{\mathrm{T}}$, $\mathbf{u}_3 = [0 \quad 1 \quad 1]^{\mathrm{T}}$;

$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a length 3 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ and $\mathbf{u}_3$ are generalized eigenvectors of ranks 2 and 3 (respectively).

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\left[\mathbf{u}_2 + (\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_2 t\right], \quad \mathbf{x}_3(t) = e^{\lambda t}\left[\mathbf{u}_3 + (\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_3 t + (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{u}_3\frac{t^2}{2}\right]$$

$$\Phi(t) = \left[\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)\right] = e^{t}\begin{bmatrix} 8 & 5+8t & 5t+4t^2 \\ 0 & 4 & 1+4t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{t}\begin{bmatrix} 8 & 5+8t & 5t+4t^2 \\ 0 & 4 & 1+4t \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{32}\begin{bmatrix} 4 & -5 & 5 \\ 0 & 8 & -8 \\ 0 & 0 & 32 \end{bmatrix} = e^{t}\begin{bmatrix} 1 & 2t & 3t+4t^2 \\ 0 & 1 & 4t \\ 0 & 0 & 1 \end{bmatrix}$$

37. $\lambda_1 = 2$: $\mathbf{u}_1 = [1 \quad 0 \quad 0]^{\mathrm{T}}$, $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$;

$\lambda_2 = 1$: $\mathbf{u}_2 = [9 \quad -3 \quad 0]^{\mathrm{T}}$, $\mathbf{u}_3 = [10 \quad 1 \quad -1]^{\mathrm{T}}$;

$\{\mathbf{u}_2, \mathbf{u}_3\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left[\mathbf{u}_3 + (\mathbf{A} - \lambda_2\mathbf{I})\mathbf{u}_3 t\right]$$

$$\Phi(t) = \left[\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)\right] = \begin{bmatrix} e^{2t} & 9e^{t} & (10+9t)e^{t} \\ 0 & -3e^{t} & (1-3t)e^{t} \\ 0 & 0 & -e^{t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 9e^{t} & (10+9t)e^{t} \\ 0 & -3e^{t} & (1-3t)e^{t} \\ 0 & 0 & -e^{t} \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 3 & 9 & 39 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & -3e^t + 3e^{2t} & (-13-9t)e^t + 13e^{2t} \\ 0 & e^t & 3te^t \\ 0 & 0 & e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & -3e^t + 3e^{2t} & (-13-9t)e^t + 13e^{2t} \\ 0 & e^t & 3te^t \\ 0 & 0 & e^t \end{bmatrix}$$

38. $\lambda_1 = 10: \mathbf{u}_1 = [4 \quad 1 \quad 0]^{\mathrm{T}}$, $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$;

$\lambda_2 = 5: \mathbf{u}_2 = [50 \quad 0 \quad 0]^{\mathrm{T}}$, $\mathbf{u}_3 = [0 \quad 4 \quad -1]^{\mathrm{T}}$;

$\{\mathbf{u}_2, \mathbf{u}_3\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left[\mathbf{u}_3 + (\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{u}_3 t\right]$$

$$\Phi(t) = \left[\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)\right] = \begin{bmatrix} 4e^{10t} & 50e^{5t} & 50te^{5t} \\ e^{10t} & 0 & 4e^{5t} \\ 0 & 0 & -e^{5t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 4e^{10t} & 50e^{5t} & 50te^{5t} \\ e^{10t} & 0 & 4e^{5t} \\ 0 & 0 & -e^{5t} \end{bmatrix} \cdot \frac{1}{50}\begin{bmatrix} 0 & 50 & 200 \\ 1 & -4 & -16 \\ 0 & 0 & -50 \end{bmatrix} = \begin{bmatrix} e^{5t} & 4e^{10t} - 4e^{5t} & 16e^{10t} - (16+50t)e^{5t} \\ 0 & e^{10t} & 4e^{10t} - 4e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$$

39. $\lambda_2 = 1: \mathbf{u}_1 = [3 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$, $\mathbf{u}_2 = [0 \quad 1 \quad 0 \quad 0]^{\mathrm{T}}$;

$\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda_1 t}\left[\mathbf{u}_2 + (\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{u}_2 t\right]$$

$\lambda_2 = 2: \mathbf{u}_3 = [144 \quad 36 \quad 12 \quad 0]^{\mathrm{T}}$, $\mathbf{u}_4 = [0 \quad 27 \quad 17 \quad 4]^{\mathrm{T}}$;

$\{\mathbf{u}_3, \mathbf{u}_4\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_3$, so $\mathbf{u}_4$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_3(t) = e^{\lambda_2 t}\mathbf{u}_3, \quad \mathbf{x}_4(t) = e^{\lambda_2 t}\left[\mathbf{u}_4 + (\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{u}_4 t\right]$$

$$\Phi(t) = \left[\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t) \quad \mathbf{x}_4(t)\right] = \begin{bmatrix} 3e^t & 3te^t & 144e^{2t} & 144te^{2t} \\ 0 & e^t & 36e^{2t} & (27+36t)e^{2t} \\ 0 & 0 & 12e^{2t} & (17+12t)e^{2t} \\ 0 & 0 & 0 & 4e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^t & 3te^t & 144e^{2t} & 144te^{2t} \\ 0 & e^t & 36e^{2t} & (27+36t)e^{2t} \\ 0 & 0 & 12e^{2t} & (17+12t)e^{2t} \\ 0 & 0 & 0 & 4e^{2t} \end{bmatrix} \cdot \frac{1}{48} \begin{bmatrix} 16 & 0 & -192 & 816 \\ 0 & 48 & -144 & 288 \\ 0 & 0 & 4 & -17 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 3te^t & (-12-9t)e^t + 12te^{2t} & (51+18t)e^t + (-51+36t)e^{2t} \\ 0 & e^t & -3e^t + 3e^{2t} & 6e^t + (-6+9t)e^{2t} \\ 0 & 0 & e^{2t} & 3te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}$$

40. $\lambda_1 = 3:\ \mathbf{u}_1 = \begin{bmatrix}100 & 20 & 4 & 1\end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$;

$\lambda = 2:\ \mathbf{u}_2 = \begin{bmatrix}16 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$, $\mathbf{u}_3 = \begin{bmatrix}0 & 4 & 0 & 0\end{bmatrix}^{\mathrm{T}}$, $\mathbf{u}_4 = \begin{bmatrix}0 & -1 & 1 & 0\end{bmatrix}^{\mathrm{T}}$;

$\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a length 3 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ and $\mathbf{u}_4$ are generalized eigenvectors of ranks 2 and 3 (respectively).

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left[\mathbf{u}_3 + (\mathbf{A} - \lambda_2\mathbf{I})\mathbf{u}_3 t\right],$$

$$\mathbf{x}_4(t) = e^{\lambda_2 t}\left[\mathbf{u}_4 + (\mathbf{A} - \lambda_2\mathbf{I})\mathbf{u}_4 t + (\mathbf{A} - \lambda_2\mathbf{I})^2\mathbf{u}_4 \frac{t^2}{2}\right]$$

$$\Phi(t) = \begin{bmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) & \mathbf{x}_4(t)\end{bmatrix} = \begin{bmatrix} 100e^{3t} & 16e^{2t} & 16te^{2t} & 8t^2e^{2t} \\ 20e^{3t} & 0 & 4e^{2t} & (-1+4t)e^{2t} \\ 4e^{3t} & 0 & 0 & e^{2t} \\ e^{3t} & 0 & 0 & 0 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 100e^{3t} & 16e^{2t} & 16te^{2t} & 8t^2e^{2t} \\ 20e^{3t} & 0 & 4e^{2t} & (-1+4t)e^{2t} \\ 4e^{3t} & 0 & 0 & e^{2t} \\ e^{3t} & 0 & 0 & 0 \end{bmatrix} \cdot \frac{1}{16} \begin{bmatrix} 0 & 0 & 0 & 16 \\ 1 & 0 & 0 & -100 \\ 0 & 4 & 4 & -96 \\ 0 & 0 & 16 & -64 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 4te^{2t} & (4t+8t^2)e^{2t} & 100e^{3t} - (100+96t+32t^2)e^{2t} \\ 0 & e^{2t} & 4te^{2t} & 20e^{3t} - (20+16t)e^{2t} \\ 0 & 0 & e^{2t} & 4e^{3t} - 4e^{2t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

SECTION 5.7

NONHOMOGENEOUS LINEAR SYSTEMS

1. Substitution of the trial solution $x_p(t) = a$, $y_p(t) = b$ yields the equations

$$a + 2b + 3 = 0, \quad 2a + b - 2 = 0$$

with solution $a = \frac{7}{3}$, $b = -\frac{8}{3}$. Thus we obtain the particular solution

$$x(t) = \frac{7}{3}, \quad y(t) = -\frac{8}{3}.$$

2. When we substitute the trial solution $x_p(t) = a_1 + b_1 t$, $y_p(t) = a_2 + b_2 t$ and collect coefficients, we get the equations

$$\begin{aligned} 2b_1 + b_2 &= 2 \\ 2b_1 + 3b_2 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 2a_1 + a_2 &= b_2 \\ 2a_1 + 3a_2 + 5 &= b_1 \end{aligned}.$$

We solve the first pair for $b_1 = \frac{3}{2}$ and $b_2 = -1$ Then we can solve the second pair for $a_1 = \frac{1}{8}$ and $a_2 = -\frac{5}{4}$. This gives the particular solution

$$x(t) = \frac{1}{8}(1 + 12t), \quad y(t) = -\frac{1}{4}(5 + 4t).$$

3. When we substitute the trial solution

$$x_p = a_1 + b_1 t + c_1 t^2, \quad y_p = a_2 + b_2 t + c_2 t^2$$

and collect coefficients, we get the equations

$$\begin{aligned} 3a_1 + 4a_2 &= b_1 & 3b_1 + 4b_2 &= 2c_1 & 3c_1 + 4c_2 &= 0 \\ 3a_1 + 2a_2 &= b_2 & 3b_1 + 2b_2 &= 2c_2 & 3c_1 + 2c_2 + 1 &= 0 \end{aligned}$$

Working backwards, we solve first for $c_1 = -\frac{2}{3}$ and $c_2 = \frac{1}{2}$, then for $b_1 = \frac{10}{9}$ and $b_2 = -\frac{7}{6}$, and finally for $a_1 = -\frac{31}{27}$ and $a_2 = \frac{41}{36}$. This determines the particular solution $x_p(t)$, $y_p(t)$. Next, the coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 6$, with eigenvectors $\mathbf{v}_1 = [1 \quad -1]^T$ and $\mathbf{v}_2 = [4 \quad 3]^T$, respectively, so the complementary solution is given by

$$x_c(t) = c_1 e^{-t} + 4c_2 e^{6t}, \quad y_c(t) = -c_1 e^{-t} + 3c_2 e^{6t}.$$

When we impose the initial conditions $x(0)=0$, $y(0)=0$ on the general solution $x(t)=x_c(t)+x_p(t)$, $y(t)=y_c(t)+y_p(t)$ we find that $c_1=\frac{8}{7}$ and $c_2=\frac{1}{756}$. This finally gives the desired particular solution

$$x(t)=\frac{1}{756}\left(864e^{-t}+4e^{6t}-868+840t-504t^2\right),$$
$$y(t)=\frac{1}{756}\left(-864e^{-t}+3e^{6t}+861-882t+378t^2\right).$$

4. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1=-5$ and $\lambda_2=-2$, with eigenvectors $\mathbf{v}_1=\begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$ and $\mathbf{v}_2=\begin{bmatrix}1 & -6\end{bmatrix}^{\mathrm{T}}$, respectively, so the complementary solution is given by

$$x_c(t)=c_1e^{5t}+c_2e^{-2t},\quad y_c(t)=c_1e^{5t}-6c_2e^{-2t}.$$

Then we try $x_p(t)=ae^t$, $y_p(t)=be^t$ and find readily the particular solution $x_p(t)=-\frac{1}{12}e^t$, $y_p(t)=-\frac{3}{4}e^t$. Thus the general solution is

$$x(t)=c_1e^{5t}+c_2e^{-2t}-\frac{1}{12}e^t,\quad y(t)=c_1e^{5t}-6c_2e^{-2t}-\frac{3}{4}e^t.$$

Finally we apply the initial conditions $x(0)=y(0)=1$ to determine $c_1=\frac{33}{28}$ and $c_2=-\frac{2}{21}$. The resulting particular solution is given by

$$x(t)=\frac{1}{84}\left(99e^{5t}-8e^{-2t}-7e^t\right),\quad y(t)=\frac{1}{84}\left(99e^{5t}+48e^{-2t}-63e^t\right).$$

5. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1=-1$ and $\lambda_2=5$, so the nonhomogeneous term e^{-t} duplicates part of the complementary solution. We therefore try the particular solution

$$x_p(t)=a_1+b_1e^{-t}+c_1te^{-t},\quad y_p(t)=a_2+b_2e^{-t}+c_2te^{-t}.$$

Upon solving the six linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we obtain the particular solution

$$x(t)=\frac{1}{3}\left(-12-e^{-t}-7te^{-t}\right),\quad y(t)=\frac{1}{3}\left(-6-7te^{-t}\right).$$

6. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \frac{1}{2}\left(7 \pm \sqrt{89}\right)$, so there is no duplication. We therefore try the particular solution

$$x_p(t) = b_1 e^t + c_1 t e^t, \quad y_p(t) = b_2 e^t + c_2 t e^t .$$

Upon solving the four linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we obtain the particular solution

$$x(t) = -\frac{1}{256}(91 + 16t)e^t, \quad y(t) = \frac{1}{32}(25 + 16t)e^t .$$

7. First we try the particular solution

$$x_p(t) = a_1 \sin t + b_1 \cos t, \quad y_p(t) = a_2 \sin t + b_2 \cos t .$$

Upon solving the four linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we find that $a_1 = -\frac{21}{82}$, $b_1 = -\frac{25}{82}$, $a_2 = -\frac{15}{41}$, and $b_2 = -\frac{12}{41}$. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -9$, with eigenvectors $\mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}$ and $\mathbf{v}_2 = [2 \quad -3]^{\mathrm{T}}$, respectively, so the complementary solution is given by

$$x_c(t) = c_1 e^t + 2c_2 e^{-9t}, \quad y_c(t) = c_1 e^t - 3c_2 e^{-9t} .$$

When we impose the initial conditions $x(0) = 1$, $y(0) = 0$ we find that $c_1 = \frac{9}{10}$ and $c_2 = \frac{83}{410}$. It follows that the desired particular solution $x = x_c + x_p$, $y = y_c + y_p$ is given by

$$x(t) = \frac{1}{410}\left(369e^t + 166e^{-9t} - 125\cos t - 105\sin t\right),$$
$$y(t) = \frac{1}{410}\left(369e^t - 249e^{-9t} - 120\cos t - 150\sin t\right).$$

8. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \pm 2i$, so the complementary function involves $\cos 2t$ and $\sin 2t$. There being therefore no duplication, we substitute the trial solution

$$x_p(t) = a_1 \sin t + b_1 \cos t, \quad y_p(t) = a_2 \sin t + b_2 \cos t$$

into the given nonhomogeneous system. Upon solving the four linear equations that result upon collection of coefficients, we obtain the particular solution

$$x(t) = \frac{1}{3}(17\cos t + 2\sin t), \quad y(t) = \frac{1}{3}(3\cos t + 5\sin t).$$

9. Here the associated homogeneous system is the same as in Problem 8, so the nonhomogeneous term $\cos 2t$ duplicates the complementary function. We therefore substitute the trial solution

$$x_p(t) = a_1 \sin 2t + b_1 \cos 2t + c_1 t \sin 2t + d_1 t \cos 2t$$
$$y_p(t) = a_2 \sin 2t + b_2 \cos 2t + c_2 t \sin 2t + d_2 t \cos 2t$$

and use a computer algebra system to solve the system of 8 linear equations that results when we collect coefficients in the usual way. This gives the particular solution

$$x(t) = \frac{1}{4}(\sin 2t + 2t\cos 2t + t\sin 2t), \quad y(t) = \frac{1}{4}t\sin 2t.$$

10. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \pm i\sqrt{3}$, so there is no duplication. Substitution of the trial solution

$$x_p(t) = a_1 e^t \cos t + b_1 e^t \sin t, \quad y_p(t) = a_2 e^t \cos t + b_2 e^t \sin t$$

yields the equations

$$\begin{aligned} 2a_2 + b_1 &= 0 \\ 2b_2 - a_1 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} -2a_1 + 2a_2 + b_2 &= 0 \\ -a_2 - 2b_1 + 2b_2 &= 1 \end{aligned}.$$

The first two equations enable us to eliminate two of the variables immediately, and we readily solve for the values $a_1 = \frac{4}{13}$, $a_2 = \frac{3}{13}$, $b_1 = -\frac{6}{13}$, and $b_2 = \frac{2}{13}$ that give the particular solution

$$x(t) = \frac{1}{13}e^t(4\cos t - 6\sin t), \quad y(t) = \frac{1}{13}e^t(3\cos t + 2\sin t).$$

11. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$, so there is duplication of constant terms. We therefore substitute the particular solution

$$x_p(t) = a_1 + b_1 t, \quad y_p(t) = a_2 + b_2 t$$

and solve the resulting equations for $a_1 = -2$, $a_2 = 0$, $b_1 = -2$, and $b_2 = 1$. The eigenvectors of the coefficient matrix associated with the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$ are $\mathbf{v}_1 = [2 \quad -1]^T$ and $\mathbf{v}_2 = [2 \quad 1]^T$, respectively, so the general solution of the given nonhomogeneous system is given by

$$x(t) = 2c_1 + 2c_2 e^{4t} - 2 - 2t, \quad y(t) = -c_1 + c_2 e^{4t} + t.$$

When we impose the initial conditions $x(0)=1$, $y(0)=-1$ we find readily that $c_1=\frac{5}{4}$, $c_2=\frac{1}{4}$. This gives the desired particular solution

$$x(t)=\frac{1}{2}\left(1-4t+e^{4t}\right),\quad y(t)=\frac{1}{4}\left(-5+4t+e^{4t}\right).$$

12. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1=0$ and $\lambda_2=2$, so there is duplication of constant terms in the first natural attempt. We must multiply the *t*-terms by *t* and include all lower-degree terms in the trial solution. Thus we substitute the trial solution

$$x_p(t)=a_1+b_1t+c_1t^2,\quad y_p(t)=a_2+b_2t+c_2t^2.$$

The resulting six equations in the coefficients are satisfied by $a_1=b_1=a_2=b_2=0$, $c_1=1$, $c_2=-1$. This gives the particular solution

$$x(t)=t^2,\quad y(t)=-t^2.$$

13. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1=1$ and $\lambda_2=3$, so there is duplication of e^t terms. We therefore substitute the trial solution

$$x_p(t)=(a_1+b_1t)e^t,\quad y_p(t)=(a_2+b_2t)e^t$$

This leads readily to the particular solution

$$x(t)=\frac{1}{2}(1+5t)e^t,\quad y(t)=-\frac{5}{2}te^t.$$

14. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1=0$ and $\lambda_2=4$, so there is duplication of both constant terms and e^{4t} terms. We therefore substitute the particular solution

$$x_p(t)=a_1+b_1t+c_1e^{4t}+d_1te^{4t},\quad y_p(t)=a_2+b_2t+c_2e^{4t}+d_2te^{4t}.$$

When we use a computer algebra system to solve the resulting system of 8 equations in 8 unknowns, we find that a_2 and c_2 can be chosen arbitrarily. With both zero we get the particular solution

$$x(t)=\frac{1}{8}\left(-2+4t-e^{4t}+2te^{4t}\right),\quad y(t)=\frac{t}{2}\left(-2+e^{4t}\right).$$

In Problems 15 and 16 the amounts $x_1(t)$ and $x_2(t)$ in the two tanks satisfy the equations

$$x_1'=rc_0-k_1x_1,\quad x_2'=k_1x_1-k_2x_2,$$

where $k_i = \dfrac{r}{V_i}$, in terms of the flow rate r, the inflowing concentration c_0, and the volumes V_1 and V_2 of the two tanks.

15. **(a)** We solve the initial value problem

$$x_1' = 20 - \frac{x_1}{10}, \quad x_1(0) = 0$$

$$x_2' = \frac{x_1}{10} - \frac{x_2}{20}, \quad x_2(0) = 0$$

for $x_1(t) = 200\left(1 - e^{-t/10}\right)$, $x_2(t) = 400\left(1 + e^{-t/10} - 2e^{-t/20}\right)$.

(b) Evidently $x_1(t) \to 200\,\text{gal}$ and $x_2(t) \to 400\,\text{gal}$ as $t \to \infty$.

(c) It takes about 6 min 56 sec for tank 1 to reach a salt concentration of 1 lb/gal, and about 24 min 34 sec for tank 2 to reach this concentration.

16. **(a)** We solve the initial value problem

$$x_1' = 30 - \frac{x_1}{20}, \quad x_1(0) = 0$$

$$x_2' = \frac{x_1}{20} - \frac{x_2}{10}, \quad x_2(0) = 0$$

for $x_1(t) = 600\left(1 - e^{-t/20}\right)$, $x_2(t) = 300\left(1 + e^{-t/10} - 2e^{-t/20}\right)$.

(b) Evidently $x_1(t) \to 600\,\text{gal}$ and $x_2(t) \to 300\,\text{gal}$ as $t \to \infty$.

(c) It takes about 8 min 7 sec for tank 1 to reach a salt concentration of 1 lb/gal, and about 17 min 13 sec for tank 2 to reach this concentration.

In Problems 17–34 we apply the variation of parameters formula in Eq. (28) of Section 5.6. The answers shown below were actually calculated using the *Mathematica* code listed in the computing project for Section 5.7. For instance, for Problem 17 we first enter the coefficient matrix

```
A = {{6, -7}, {1, -2}};
```

the initial vector

```
x0 = {{0}, {0}};
```

and the vector

```
f[t_] := {{60}, {90}};
```

of nonhomogeneous terms. It simplifies the notation to rename *Mathematica*'s exponential matrix function by defining

```
exp[A_] := MatrixExp[A]
```

Then the integral in the variation of parameters formula is given by

```
integral = Integrate[exp[-A*s] . f[s], {s, 0, t}] // Simplify
```

and yields the output

$$\begin{bmatrix} -102+7e^{-5t}+95e^{t} \\ -96+e^{-5t}+95e^{t} \end{bmatrix}.$$

Finally the desired particular solution is given by

```
solution = exp[A*t] . (x0 + integral) // Simplify
```

which yields

$$\begin{bmatrix} 102-7e^{-5t}-95e^{t} \\ 96-e^{-5t}-95e^{t} \end{bmatrix}.$$

(Maple and MATLAB versions of this computation are provided in the applications manual that accompanies the textbook.)

In each succeeding problem, we need only substitute the given coefficient matrix **A**, initial vector **x0**, and the vector **f** of nonhomogeneous terms in the above commands, and then re-execute them in turn. We give below only the component functions of the final results.

17. $x_1(t) = 102 - 95e^{-t} - 7e^{5t}$, $x_2(t) = 96 - 95e^{-t} - e^{5t}$

18. $x_1(t) = 68 - 110t - 75e^{-t} + 7e^{5t}$, $x_2(t) = 74 - 80t - 75e^{-t} + e^{5t}$

19. $x_1(t) = -70 - 60t + 16e^{-3t} + 54e^{2t}$, $x_2(t) = 5 - 60t - 32e^{-3t} + 27e^{2t}$

20. $x_1(t) = 3e^{2t} + 60te^{2t} - 3e^{-3t}$, $x_2(t) = -6e^{2t} + 30te^{2t} + 6e^{-3t}$

21. $x_1(t) = -e^{-t} - 14e^{2t} + 15e^{3t}$, $x_2(t) = -5e^{-t} - 10e^{2t} + 15e^{3t}$

22. $x_1(t) = -10e^{-t} - 7te^{-t} + 10e^{3t} - 5te^{3t}$, $x_2(t) = -15e^{-t} - 35te^{-t} + 15e^{3t} - 5te^{3t}$

23. $x_1(t) = 3 + 11t + 8t^2$, $x_2(t) = 5 + 17t + 24t^2$

24. $x_1(t) = 2 + t + \ln t$, $x_2(t) = 5 + 3t - \dfrac{1}{t} + 3\ln t$

25. $x_1(t) = -1 + 8t + \cos t - 8\sin t$, $x_2(t) = -2 + 4t + 2\cos t - 3\sin t$

26. $x_1(t) = 3\cos t - 32\sin t + 17t\cos t + 4t\sin t$, $x_2(t) = 5\cos t - 13\sin t + 6t\cos t + 5t\sin t$

27. $x_1(t) = 8t^3 + 6t^4$, $x_2(t) = 3t^2 - 2t^3 + 3t^4$

28. $x_1(t) = -7 + 14t - 6t^2 + 4t^2 \ln t$, $x_2(t) = -7 + 9t - 3t^2 + \ln t - 2t \ln t + 2t^2 \ln t$

29. $x_1(t) = t\cos t - \ln(\cos t)\sin t$, $x_2(t) = t\sin t - \ln(\cos t)\cos t$

30. $x_1(t) = \frac{1}{2}t^2 \cos 2t$, $x_2(t) = \frac{1}{2}t^2 \sin 2t$

31. $x_1(t) = (9t^2 + 4t^3)e^t$, $x_2(t) = 6t^2 e^t$, $x_3(t) = 6te^t$

32. $x_1(t) = (44 + 18t)e^t + (-44 + 26t)e^{2t}$, $x_2(t) = 6e^t + (-6 + 6t)e^{2t}$, $x_3(t) = 2te^{2t}$

33. $x_1(t) = 15t^2 + 60t^3 + 95t^4 + 12t^5$, $x_2(t) = 15t^2 + 55t^3 + 15t^4$, $x_3(t) = 15t^2 + 20t^3$,
$x_4(t) = 15t^2$

34. $x_1(t) = 4t^3 + (4 + 16t + 8t^2)e^{2t}$, $x_2(t) = 3t^2 + (2 + 4t)e^{2t}$, $x_3(t) = (2 + 4t + 2t^2)e^{2t}$,
$x_4(t) = (1 + t)e^{2t}$