CHAPTER 7

LAPLACE TRANSFORM METHODS

SECTION 7.1

LAPLACE TRANSFORMS AND INVERSE TRANSFORMS

The objectives of this section are especially clear cut. They include familiarity with the definition of the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ that is given in Equation (1) in the textbook, the direct application of this definition to calculate Laplace transforms of simple functions (as in Examples 1–3), and the use of known transforms (those listed in Figure 7.1.2) to find Laplace transforms and inverse transforms (as in Examples 4-6). Perhaps students need to be told explicitly to memorize the transforms that are listed in the short table that appears in Figure 7.1.2.

1. $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\, dt \qquad (u = -st, \quad du = -s\, dt)$

$$= \int_0^{-\infty} \left[\frac{1}{s^2}\right] ue^u\, du = \frac{1}{s^2}\Big[(u-1)e^u\Big]_0^{-\infty} = \frac{1}{s^2}$$

2. We substitute $u = -st$ in the tabulated integral

$$\int u^2 e^u\, du = e^u\left(u^2 - 2u + 2\right) + C$$

(or, alternatively, integrate by parts) and get

$$\mathcal{L}\left\{t^2\right\} = \int_0^\infty e^{-st} t^2\, dt = \left[-e^{-st}\left(\frac{t^2}{s} + \frac{2t}{s^2} + \frac{2}{s^3}\right)\right]_{t=0}^\infty = \frac{2}{s^3}.$$

3. $\mathcal{L}\left\{e^{3t+1}\right\} = \int_0^\infty e^{-st} e^{3t+1}\, dt = e\int_0^\infty e^{-(s-3)t}\, dt = \dfrac{e}{s-3}$

4. With $a = -s$ and $b = 1$ the tabulated integral

$$\int e^{au} \cos bu\, du = e^{au}\left[\frac{a\cos bu + b\sin bu}{a^2 + b^2}\right] + C$$

yields

$$\mathcal{L}\{\cos t\} = \int_0^\infty e^{-st} \cos t\, dt = \left[\frac{e^{-st}(-s\cos t + \sin t)}{s^2 + 1}\right]_{t=0}^\infty = \frac{s}{s^2+1}.$$

5. $\mathcal{L}\{\sinh t\} = \frac{1}{2}\mathcal{L}\{e^t - e^{-t}\} = \frac{1}{2}\int_0^\infty e^{-st}\left(e^t - e^{-t}\right)dt = \frac{1}{2}\int_0^\infty \left(e^{-(s-1)t} - e^{-(s+1)t}\right)dt$

$$= \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = \frac{1}{s^2-1}$$

6. $\mathcal{L}\{\sin^2 t\} = \int_0^\infty e^{-st}\sin^2 t\,dt = \frac{1}{2}\int_0^\infty e^{-st}\left(1-\cos 2t\right)dt$

$$= \frac{1}{2}\left[e^{-st}\left(-\frac{1}{s}\right) - e^{-st}\cdot\frac{-s\cos 2t + 2\sin 2t}{s^2+4}\right]_{t=0}^{\infty} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

7. $\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^1 = \frac{1-e^{-s}}{s}$

8. $\mathcal{L}\{f(t)\} = \int_1^2 e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_1^2 = \frac{e^{-s} - e^{-2s}}{s}$

9. $\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}t\,dt = \frac{1-e^{-s}-se^{-s}}{s^2}$

10. $\mathcal{L}\{f(t)\} = \int_0^1 (1-t)e^{-st}\,dt = \left[-e^{-st}\left(\frac{1}{s} - \frac{t}{s} - \frac{1}{s^2}\right)\right]_0^1 = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}$

11. $\mathcal{L}\{\sqrt{t} + 3t\} = \frac{\Gamma(3/2)}{s^{3/2}} + 3\cdot\frac{1}{s^2} = \frac{\sqrt{\pi}}{2s^{3/2}} + \frac{3}{s^2}$

12. $\mathcal{L}\{3t^{5/2} - 4t^3\} = 3\cdot\frac{\Gamma(7/2)}{s^{7/2}} - 4\cdot\frac{3!}{s^4} = \frac{45\sqrt{\pi}}{8s^{7/2}} - \frac{24}{s^2}$

13. $\mathcal{L}\{t - 2e^{3t}\} = \frac{1}{s^2} - \frac{2}{s-3}$

14. $\mathcal{L}\{t^{3/2} + e^{-10t}\} = \frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{s+10} = \frac{3\sqrt{\pi}}{4s^{5/2}} + \frac{1}{s+10}$

15. $\mathcal{L}\{1 + \cosh 5t\} = \frac{1}{s} + \frac{s}{s^2-25}$

16. $\mathcal{L}\{\sin 2t + \cos 2t\} = \dfrac{2}{s^2+4}+\dfrac{s}{s^2+4} = \dfrac{s+2}{s^2+4}$

17. $\mathcal{L}\{\cos^2 2t\} = \dfrac{1}{2}\mathcal{L}\{1+\cos 4t\} = \dfrac{1}{2}\left(\dfrac{1}{s}+\dfrac{s}{s^2+16}\right)$

18. $\mathcal{L}\{\sin 3t\cos 3t\} = \dfrac{1}{2}\mathcal{L}\{\sin 6t\} = \dfrac{1}{2}\cdot\dfrac{6}{s^2+36} = \dfrac{3}{s^2+36}$

19. $\mathcal{L}\{(1+t)^3\} = \mathcal{L}\{1+3t+3t^2+t^3\} = \dfrac{1}{s}+3\cdot\dfrac{1!}{s^2}+3\cdot\dfrac{2!}{s^3}+\dfrac{3!}{s^4} = \dfrac{1}{s}+\dfrac{3}{s^2}+\dfrac{6}{s^3}+\dfrac{6}{s^4}$

20. Integrating by parts with $u = t,\ dv = e^{-(s-1)t}dt,$ we get

$$\mathcal{L}\{te^t\} = \int_0^\infty e^{-st}\,te^t\,dt = \int_0^\infty te^{-(s-1)t}\,dt$$
$$= \left[\frac{-te^{-(s-1)t}}{s-1}\right]_0^\infty + \frac{1}{s-1}\int_0^\infty e^{-st}e^t\,dt = \frac{1}{s-1}\mathcal{L}\{t\} = \frac{1}{(s-1)^2}.$$

21. Integration by parts with $u = t$ and $dv = e^{-st}\cos 2t\,dt$ yields

$$\mathcal{L}\{t\cos 2t\} = \int_0^\infty te^{-st}\cos 2t\,dt = -\frac{1}{s^2+4}\int_0^\infty e^{-st}\left(-s\cos 2t+2\sin 2t\right)dt$$
$$= -\frac{1}{s^2+4}\left[-s\,\mathcal{L}\{\cos 2t\}+2\,\mathcal{L}\{\sin 2t\}\right]$$
$$= -\frac{1}{s^2+4}\left[\frac{-s^2}{s^2+4}+\frac{4}{s^2+4}\right] = \frac{s^2-4}{\left(s^2+4\right)^2}.$$

22. $\mathcal{L}\{\sinh^2 3t\} = \dfrac{1}{2}\mathcal{L}\{\cosh 6t-1\} = \dfrac{1}{2}\left(\dfrac{s}{s^2-36}-\dfrac{1}{s}\right)$

23. $\mathcal{L}^{-1}\left\{\dfrac{3}{s^4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{6}{s^4}\right\} = \dfrac{1}{2}t^3$

24. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{3/2}}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2}{\sqrt{\pi}}\cdot\dfrac{\sqrt{\pi}}{2s^{3/2}}\right\} = \dfrac{2}{\sqrt{\pi}}\cdot t^{1/2} = 2\sqrt{\dfrac{t}{\pi}}$

25. $\mathcal{L}^{-1}\left\{\dfrac{1}{s}-\dfrac{2}{s^{5/2}}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s}-\dfrac{2}{\Gamma(5/2)}\cdot\dfrac{\Gamma(5/2)}{s^{5/2}}\right\} = 1-\dfrac{2}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\cdot t^{3/2} = 1-\dfrac{8t^{3/2}}{3\sqrt{\pi}}$

26. $\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = = e^{-5t}$

27. $\mathcal{L}^{-1}\left\{\frac{3}{s-4}\right\} = 3\cdot\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = 3e^{4t}$

28. $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}\right\} = 3\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{1}{2}\cdot\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = 3\cos 2t+\frac{1}{2}\sin 2t$

29. $\mathcal{L}^{-1}\left\{\frac{5-3s}{s^2+9}\right\} = \frac{5}{3}\cdot\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}-3\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \frac{5}{3}\sin 3t-3\cos 3t$

30. $\mathcal{L}^{-1}\left\{\frac{9+s}{4-s^2}\right\} = -\frac{9}{2}\cdot\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\}-\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} = -\frac{9}{2}\sinh 2t-\cosh 2t$

31. $\mathcal{L}^{-1}\left\{\frac{10s-3}{25-s^2}\right\} = -10\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2-25}\right\}+\frac{3}{5}\cdot\mathcal{L}^{-1}\left\{\frac{5}{s^2-25}\right\} = -10\cosh 5t+\frac{3}{5}\sinh 5t$

32. $\mathcal{L}^{-1}\left\{2\cdot\frac{e^{-3s}}{s}\right\} = 2u(t-3) = 2u_3(t)$ [See Example 8 in the textbook.]

33. $\mathcal{L}\{\sin kt\} = \mathcal{L}\left\{\frac{e^{ikt}-e^{-ikt}}{2i}\right\} = \frac{1}{2i}\left(\frac{1}{s-ik}-\frac{1}{s+ik}\right)$

$= \frac{1}{2i}\cdot\frac{2ik}{(s-ik)(s-ik)} = \frac{k}{s^2+k^2}$ (because $i^2=-1$)

34. $\mathcal{L}\{\sinh kt\} = \mathcal{L}\left\{\frac{e^{kt}-e^{-kt}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s-k}-\frac{1}{s+k}\right) = \frac{1}{2}\cdot\frac{2k}{s^2-k^2} = \frac{k}{s^2-k^2}$

35. Using the given tabulated integral with $a=-s$ and $b=k$, we find that

$$\mathcal{L}\{\cos kt\} = \int_0^\infty e^{-st}\cos kt\,dt = \left[\frac{e^{-st}}{s^2+k^2}\left(-s\cos kt+k\sin kt\right)\right]_{t=0}^{\infty}$$

$$= \lim_{t\to\infty}\left(\frac{e^{-st}}{s^2+k^2}\left(-s\cos kt+k\sin kt\right)\right)-\frac{e^0}{s^2+k^2}(-s\cdot 1+k\cdot 0) = \frac{s}{s^2+k^2}.$$

36. Evidently the function $f(t) = \sin(e^{t^2})$ is of exponential order because it is bounded; we can simply take $c = 0$ and $M = 1$ in Eq. (23) of this section in the text. However, its derivative $f'(t) = 2t\,e^{t^2}\cos(e^{t^2})$ is *not* bounded by any exponential function e^{ct}, because $e^{t^2}/e^{ct} = e^{t^2-ct} \to \infty$ as $t \to \infty$.

37. $f(t) = 1 - u_a(t) = 1 - u(t - a)$ so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{u_a(t)\} = \frac{1}{s} - \frac{e^{-as}}{s} = s^{-1}(1 - e^{-as}).$$

For the graph of f, note that $f(a) = 1 - u(a) = 1 - 1 = 0.$

38. $f(t) = u(t-a) - u(t-b)$, so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_a(t)\} - \mathcal{L}\{u_b(t)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = s^{-1}\left(e^{-as} - e^{-bs}\right).$$

For the graph of f, note that $f(a) = u(0) - u(a-b) = 1 - 0 = 1$ because $a < b$, but $f(b) = u(b-a) - u(0) = 1 - 1 = 0.$

39. Use of the geometric series gives

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty}\mathcal{L}\{u(t-n)\} = \sum_{n=0}^{\infty}\frac{e^{-ns}}{s} = \frac{1}{s}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots\right)$$

$$= \frac{1}{s}\left(1 + (e^{-s}) + (e^{-s})^2 + (e^{-s})^3 + \cdots\right) = \frac{1}{s}\cdot\frac{1}{1-e^{-s}} = \frac{1}{s\left(1-e^{-s}\right)}.$$

40. Use of the geometric series gives

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty}(-1)^n\mathcal{L}\{u(t-n)\} = \sum_{n=0}^{\infty}\frac{(-1)^n e^{-ns}}{s} = \frac{1}{s}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right)$$

$$= \frac{1}{s}\left(1 + (-e^{-s}) + (-e^{-s})^2 + (-e^{-s})^3 + \cdots\right) = \frac{1}{s}\cdot\frac{1}{1-(-e^{-s})} = \frac{1}{s\left(1+e^{-s}\right)}.$$

41. By checking values at sample points, you can verify that $g(t) = 2f(t) - 1$ in terms of the square wave function $f(t)$ of Problem 40. Hence

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{2f(t)-1\} = \frac{2}{s\left(1+e^{-s}\right)} - \frac{1}{s} = \frac{1}{s}\left(\frac{2}{1+e^{-s}} - 1\right) = \frac{1}{s}\cdot\frac{1-e^{-s}}{1+e^{-s}}$$

$$= \frac{1}{s}\cdot\frac{1-e^{-s}}{1+e^{-s}}\cdot\frac{e^{s/2}}{e^{s/2}} = \frac{1}{s}\cdot\frac{e^{s/2}-e^{-s/2}}{e^{s/2}+e^{-s/2}} = \frac{1}{s}\cdot\frac{\frac{1}{2}\left(e^{s/2}-e^{-s/2}\right)}{\frac{1}{2}\left(e^{s/2}+e^{-s/2}\right)}$$

$$= \frac{1}{s} \cdot \frac{\sinh(s/2)}{\cosh(s/2)} = \frac{1}{s} \tanh \frac{s}{2}.$$

42. Let's refer to $(n-1, n]$ as an odd interval if the integer n is odd, and even interval if n is even. Then our function $h(t)$ has the value a on odd intervals, the value b on even intervals. Now the unit step function $f(t)$ of Problem 40 has the value 1 on odd intervals, the value 0 on even intervals. Hence the function $(a-b)f(t)$ has the value $(a-b)$ on odd intervals, the value 0 on even intervals. Finally, the function $(a-b)f(t)+b$ has the value $(a-b)+b=a$ on odd intervals, the value b on even intervals, and hence $(a-b)f(t)+b=h(t)$. Therefore

$$L\{h(t)\} = L\{(a-b)f(t)\} + L\{b\} = \frac{a-b}{s(1+e^{-s})} + \frac{b}{s} = \frac{a+be^{-s}}{s(1+e^{-s})}.$$

SECTION 7.2

TRANSFORMATION OF INITIAL VALUE PROBLEMS

The focus of this section is on the use of transforms of derivatives (Theorem 1) to solve initial value problems (as in Examples 1 and 2). Transforms of integrals (Theorem 2) appear less frequently in practice, and the extension of Theorem 1 at the end of Section 7.2 may be considered entirely optional (except perhaps for electrical engineering students).

In Problems 1–10 we give first the transformed differential equation, then the transform $X(s)$ of the solution, and finally the inverse transform $x(t)$ of $X(s)$.

1. $[s^2X(s) - 5s] + 4\{X(s)\} = 0$

$$X(s) = \frac{5s}{s^2+4} = 5 \cdot \frac{s}{s^2+4}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 5\cos 2t$$

2. $[s^2X(s) - 3s - 4] + 9[X(s)] = 0$

$$X(s) = \frac{3s+4}{s^2+9} = 3 \cdot \frac{s}{s^2+9} + \frac{4}{3} \cdot \frac{3}{s^2+9}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 3\cos 3t + (4/3)\sin 3t$$

3. $[s^2X(s) - 2] - [sX(s)] - 2[X(s)] = 0$

$$X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s-2)(s+1)} = \frac{2}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

$x(t) = (2/3)(e^{2t} - e^{-t})$

4. $[s^2X(s) - 2s + 3] + 8[s\,X(s) - 2] + 15[X(s)] = 0$

$$X(s) = \frac{2s+13}{s^2+8s+15} = \frac{7}{2}\cdot\frac{1}{s+3} - \frac{3}{2}\cdot\frac{1}{s+5}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (7/2)e^{-3t} - (3/2)e^{-5t}$

5. $[s^2X(s)] + [X(s)] = 2/(s^2+4)$

$$X(s) = \frac{2}{(s^2+1)(s^2+4)} = \frac{2}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\cdot\frac{2}{s^2+4}$$

$x(t) = (2\sin t - \sin 2t)/3$

6. $[s^2X(s)] + 4[X(s)] = \mathcal{L}\{\cos t\} = s/(s^2+1)$

$$X(s) = \frac{2}{(s^2+1)(s^2+4)} = \frac{1}{3}\cdot\frac{s}{s^2+1} - \frac{1}{3}\cdot\frac{s}{s^2+4}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (\cos t - \cos 2t)/3$

7. $[s^2X(s) - s] + [X(s)] = s/s^2 + 9)$

$(s^2+1)X(s) = s + s/(s^2+9) = (s^3+10s)/(s^2+9)$

$$X(s) = \frac{s^2+10s}{(s^2+1)(s^2+9)} = \frac{9}{9}\cdot\frac{s}{s^2+1} - \frac{1}{8}\cdot\frac{s}{s^2+9}$$

$x(t) = (9\cos t - \cos 3t)/8$

8. $[s^2X(s)] + 9[X(s)] = \mathcal{L}\{1\} = 1/s$

$$X(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\cdot\frac{1}{s} - \frac{1}{9}\cdot\frac{s}{s^2+9}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (1 - \cos 3t)/9$

9. $s^2X(s) + 4sX(s) + 3X(s) = 1/s$

$$X(s) = \frac{1}{s(s^2+4s+3)} = \frac{1}{s(s+1)(s+3)} = \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{6}\cdot\frac{1}{s+3}$$

$x(t) = (2 - 3e^{-t} + e^{-3t})/6$

10. $[s^2X(s) - 2] + 3[sX(s)] + 2[X(s)] = \mathcal{L}\{t\} = 1/s^2$

$(s^2 + 3s + 2)X(s) = 2 + 1/s^2 = (2s^2 + 1)/s^2$

$$X(s) = \frac{2s^2+1}{s^2(s^2+3s+2)} = \frac{2s^2+1}{s^2(s+1)(s+2)} = -\frac{3}{4}\cdot\frac{1}{s}+\frac{1}{2}\cdot\frac{1}{s^2}+3\cdot\frac{1}{s+1}-\frac{9}{4}\cdot\frac{1}{s+2}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (-3 + 2t + 12e^{-t} - 9e^{-2t})/4$

11. The transformed equations are

$$sX(s) - 1 = 2X(s) + Y(s)$$
$$sY(s) + 2 = 6X(s) + 3Y(s).$$

We solve for the Laplace transforms

$$X(s) = \frac{s-5}{s(s-5)} = \frac{1}{s}$$

$$Y(s) = X(s) = \frac{-2s+10}{s(s-5)} = -\frac{2}{s}.$$

Hence the solution is given by

$$x(t) = 1, \qquad y(t) = -2.$$

12. The transformed equations are

$$s\,X(s) = X(s) + 2Y(s)$$
$$s\,Y(s) = X(s) + 1/(s+1),$$

which we solve for

$$X(s) = \frac{2}{(s-2)(s+1)^2} = \frac{2}{9}\left(\frac{1}{s-2}-\frac{1}{s+1}-3\cdot\frac{1}{(s+1)^2}\right)$$

$$Y(s) = \frac{s-1}{(s-2)(s+1)^2} = \frac{1}{9}\left(\frac{1}{s-2}-\frac{1}{s+1}-6\cdot\frac{1}{(s+1)^2}\right).$$

Hence the solution is

$$x(t) = (2/9)(e^{2t} - e^{-t} - 3t\,e^{-t})$$
$$y(t) = (1/9)(e^{2t} - e^{-t} + 6t\,e^{-t}).$$

13. The transformed equations are

$$\begin{aligned} sX(s) + 2[sY(s) - 1] + X(s) &= 0 \\ sX(s) - [sY(s) - 1] + Y(s) &= 0, \end{aligned}$$

which we solve for the transforms

$$X(s) = -\frac{2}{3s^2-1} = -\frac{2}{3}\cdot\frac{1}{s^2-1/3} = -\frac{2}{\sqrt{3}}\cdot\frac{1/\sqrt{3}}{s^2-\left(1/\sqrt{3}\right)^2}$$

$$Y(s) = \frac{3s+1}{3s^2-1} = \frac{s+1/3}{s^2-1/3} = \frac{s}{s^2-\left(1/\sqrt{3}\right)^2} + \frac{1}{\sqrt{3}}\cdot\frac{1/\sqrt{3}}{s^2-\left(1/\sqrt{3}\right)^2}.$$

Hence the solution is

$$\begin{aligned} x(t) &= -\left(2/\sqrt{3}\right)\sinh\left(t/\sqrt{3}\right) \\ y(t) &= \cosh\left(t/\sqrt{3}\right) + \left(1/\sqrt{3}\right)\sinh\left(t/\sqrt{3}\right). \end{aligned}$$

14. The transformed equations are

$$\begin{aligned} s^2X(s) + 1 + 2X(s) + 4Y(s) &= 0 \\ s^2Y(s) + 1 + X(s) + 2Y(s) &= 0, \end{aligned}$$

which we solve for

$$X(s) = \frac{-s^2+2}{s^2(s^2+4)} = \frac{1}{4}\left(2\cdot\frac{1}{s^2} - 3\cdot\frac{2}{s^2+4}\right)$$

$$Y(s) = \frac{-s^2-1}{s^2(s^2+4)} = -\frac{1}{8}\left(2\cdot\frac{1}{s^2} + 3\cdot\frac{2}{s^2+4}\right).$$

Hence the solution is

$$\begin{aligned} x(t) &= (1/4)(2t - 3\sin 2t) \\ y(t) &= (-1/8)(2t + 3\sin 2t). \end{aligned}$$

15. The transformed equations are

$$\begin{aligned} [s^2X - s] + [sX - 1] + [sY - 1] + 2X - Y &= 0 \\ [s^2Y - s] + [sX - 1] + [sY - 1] + 4X - 2Y &= 0, \end{aligned}$$

which we solve for

$$X(s) = \frac{s^2+3s+2}{s^3+3s^2+3s} = \frac{1}{3}\left(\frac{2}{s}+\frac{s+3}{s^2+3s+3}\right) = \frac{1}{3}\left(\frac{2}{s}+\frac{s+3}{(s+3/2)^2+(3/4)}\right)$$

$$= \frac{1}{3}\left(\frac{2}{s}+\frac{s+3/2}{(s+3/2)^2+(\sqrt{3}/2)^2}+\sqrt{3}\cdot\frac{\sqrt{3}/2}{(s+3/2)^2+(\sqrt{3}/2)^2}\right)$$

$$Y(s) = \frac{-s^3-2s^2+2s+4}{s^3+3s^2+3s} = \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+\frac{2s+15}{s^2+3s+3}\right)$$

$$= \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+\frac{2s+15}{(s+3/2)^2+3/4}\right)$$

$$= \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+2\cdot\frac{s+3/2}{(s+3/2)^2+(\sqrt{3}/2)^2}+8\sqrt{3}\cdot\frac{\sqrt{3}/2}{(s+3/2)^2+(\sqrt{3}/2)^2}\right).$$

Here we've used some fairly heavy-duty partial fractions (Section 7.3). The transforms

$$\mathcal{L}\left\{e^{at}\cos kt\right\} = \frac{s-a}{(s-a)^2+k^2}, \qquad \mathcal{L}\left\{e^{at}\sin kt\right\} = \frac{k}{(s-a)^2+k^2}$$

from the inside-front-cover table (with $a=-3/2,\ k=\sqrt{3}/2$) finally yield

$$x(t) = \frac{1}{3}\left\{2+e^{-3t/2}\left[\cos\left(\sqrt{3}t/2\right)+\sqrt{3}\sin\left(\sqrt{3}t/2\right)\right]\right\}$$

$$y(t) = \frac{1}{21}\left\{28-9e^t+e^{-3t/2}\left[2\cos\left(\sqrt{3}t/2\right)+8\sqrt{3}\sin\left(\sqrt{3}t/2\right)\right]\right\}.$$

16. The transformed equations are

$$s\,X(s) - 1 = X(s) + Z(s)$$

$$s\,Y(s) = X(s) + Y(s)$$

$$s\,Z(s) = -2X(s) - Z(s),$$

which we solve for

$$X(s) = \frac{s^2-1}{(s-1)(s^2+1)} = \frac{s+1}{s^2+1}$$

$$Y(s) = \frac{s+1}{(s-1)(s^2+1)} = \frac{1}{s-1}-\frac{s}{s^2+1}$$

$$Z(s) = \frac{-2s+2}{(s-1)(s^2+1)} = -\frac{2}{s^2+1}.$$

Hence the solution is

$$x(t) = \cos t + \sin t$$
$$y(t) = e^t - \cos t$$
$$z(t) = -2 \sin t.$$

17. $$f(t) = \int_0^t e^{3\tau}\, d\tau = \left[\frac{1}{3}e^{3\tau}\right]_{\tau=0}^{t} = \frac{1}{3}\left(e^{3t} - 1\right)$$

18. $$f(t) = \int_0^t 3e^{-5\tau}\, d\tau = \left[-\frac{3}{5}e^{-5\tau}\right]_{\tau=0}^{t} = \frac{3}{5}\left(1 - e^{-5t}\right)$$

19. $$f(t) = \int_0^t \tfrac{1}{2}\sin 2\tau\, d\tau = \left[-\frac{1}{4}\cos 2\tau\right]_{\tau=0}^{t} = \frac{1}{4}\left(1 - \cos 2t\right)$$

20. $$f(t) = \int_0^t \left(2\cos 3\tau + \tfrac{1}{3}\sin 3\tau\right) d\tau = \left[\frac{2}{3}\sin 3\tau - \frac{1}{9}\cos 3\tau\right]_{\tau=0}^{t} = \frac{1}{9}\left(6\sin 3t - \cos 3t + 1\right)$$

21. $$f(t) = \int_0^t \left[\int_0^\tau \sin t\, dt\right] d\tau = \int_0^t (1 - \cos\tau)\, d\tau = \left[\tau - \sin\tau\right]_{\tau=0}^{t} = t - \sin t$$

22. $$f(t) = \int_0^t \tfrac{1}{3}\sinh 3\tau\, d\tau = \left[\frac{1}{9}\cosh 3\tau\right]_{\tau=0}^{t} = \frac{1}{9}\left(\cosh 3t - 1\right)$$

23. $$f(t) = \int_0^t \left[\int_0^\tau \sinh t\, dt\right] d\tau = \int_0^t (\cosh\tau - 1)\, d\tau = \left[\sinh\tau - \tau\right]_{\tau=0}^{t} = \sinh t - t$$

24. $$f(t) = \int_0^t \left(e^{-\tau} - e^{-2\tau}\right) d\tau = \left[-e^{-\tau} + \frac{1}{2}e^{-2\tau}\right]_{\tau=0}^{t} = \frac{1}{2}\left(e^{-2t} - 2e^{-t} + 1\right)$$

25. With $f(t) = \cos kt$ and $F(s) = s/(s^2 + k^2)$, Theorem 1 in this section yields

$$\mathcal{L}\{-k \sin kt\} = \mathcal{L}\{f'(t)\} = sF(s) - 1 = s \cdot \frac{s}{s^2 + k^2} - 1 = -\frac{k^2}{s^2 + k^2},$$

so division by $-k$ yields $\mathcal{L}\{\sin kt\} = k/(s^2 + k^2)$.

26. With $f(t) = \sinh kt$ and $F(s) = k/(s^2 - k^2)$, Theorem 1 yields

$$\mathcal{L}\{f'(t)\} = \mathcal{L}\{k \cosh kt\} = ks/(s^2 - k^2) = sF(s),$$

so it follows upon division by k that $\mathcal{L}\{\cosh kt\} = s/(s^2 - k^2)$.

27. **(a)** With $f(t) = t^n e^{at}$ and $f'(t) = nt^{n-1}e^{at} + at^n e^{at}$, Theorem 1 yields

$$\mathcal{L}\{nt^{n-1}e^{at} + at^n e^{at}\} = s\,\mathcal{L}\{t^n e^{at}\}$$

so

$$n\,\mathcal{L}\{t^{n-1}e^{at}\} = (s - a)\mathcal{L}\{t^n e^{at}\}$$

and hence

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n}{s-a}\mathcal{L}\left\{t^{n-1}e^{at}\right\}.$$

(b) $n=1$: $\mathcal{L}\left\{t\,e^{at}\right\} = \dfrac{1}{s-a}\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}\cdot\dfrac{1}{s-a} = \dfrac{1}{(s-a)^2}$

$n=2$: $\mathcal{L}\left\{t^2\,e^{at}\right\} = \dfrac{2}{s-a}\mathcal{L}\left\{t\,e^{at}\right\} = \dfrac{2}{s-a}\cdot\dfrac{1}{(s-a)^2} = \dfrac{2!}{(s-a)^3}$

$n=3$: $\mathcal{L}\left\{t^3\,e^{at}\right\} = \dfrac{3}{s-a}\mathcal{L}\left\{t^2\,e^{at}\right\} = \dfrac{3}{s-a}\cdot\dfrac{2!}{(s-a)^3} = \dfrac{3!}{(s-a)^4}$

And so forth.

28. Problems 28 and 30 are the trigonometric and hyperbolic versions of essentially the same computation. For Problem 30 we let $f(t) = t\cosh kt$, so $f(0) = 0$. Then

$$f'(t) = \cosh kt + kt\sinh kt$$
$$f''(t) = 2k\sinh kt + k^2 t\cosh kt,$$

and thus $f'(0) = 1$, so Formula (5) in this section yields

$$\mathcal{L}\{2k\sinh kt + k^2 t\cosh kt\} = s^2\mathcal{L}\{t\cosh kt\} - 1,$$

$$2k\cdot\frac{k}{s^2-k^2} + k^2F(s) = s^2F(s) - 1.$$

We readily solve this last equation for

$$\mathcal{L}\{t\cosh kt\} = F(s) = \frac{s^2+k^2}{\left(s^2-k^2\right)^2}.$$

29. Let $f(t) = t\sinh kt$, so $f(0) = 0$. Then

$$f'(t) = \sinh kt + kt \cosh kt$$
$$f''(t) = 2k \cosh kt + k^2 t \sinh kt,$$

and thus $f'(0) = 0$, so Formula (5) in this section yields

$$\mathcal{L}\{2k \cosh kt + k^2 t \sinh kt\} = s^2 \mathcal{L}\{t \sinh kt\},$$

$$2k \cdot \frac{s}{s^2 - k^2} + k^2 F(s) = s^2 F(s).$$

We readily solve this last equation for

$$\mathcal{L}\{t \sinh kt\} = F(s) = \frac{2ks}{\left(s^2 - k^2\right)^2}.$$

30. See Problem 28.

31. Using the known transform of $\sin kt$ and the Problem 28 transform of $t \cos kt$, we obtain

$$\mathcal{L}\left\{\frac{1}{2k^3}\left(\sin kt - kt \cos kt\right)\right\} = \frac{1}{2k^3} \cdot \frac{k}{s^2 + k^2} - \frac{k}{2k^3} \cdot \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$$

$$= \frac{1}{2k^2}\left(\frac{1}{s^2 + k^2} - \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}\right) = \frac{1}{2k^2} \cdot \frac{2k^2}{\left(s^2 + k^2\right)^2} = \frac{1}{\left(s^2 + k^2\right)^2}$$

32. If $f(t) = u(t - a)$, then the only jump in $f(t)$ is $j_1 = 1$ at $t_1 = a$. Since $f(0) = 0$ and $f'(t) = 0$, Formula (21) in this section yields

$$0 = s\,F(\mathrm{s}) - 0 - e^{as}(1).$$

Hence $\mathcal{L}\{u(t - a)\} = F(s) = s^{-1} e^{-as}$.

33. $f(t) = u_a(t) - u_b(t) = u(t-a) - u(t-b)$, so the result of Problem 32 gives

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-a)\} - \mathcal{L}\{u(t-b)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{e^{-as} - e^{-bs}}{s}.$$

34. The square wave function of Figure 7.2.9 has a sequence $\{t_n\}$ of jumps with $t_n = n$ and $j_n = 2(-1)^n$ for $n = 1, 2, 3, \ldots$. Hence Formula (21) yields

$$0 = s\,F(s) - 1 - \sum_{n=1}^{\infty} e^{-ns} \cdot 2(-1)^n.$$

It follows that

$$
\begin{aligned}
s\,F(s) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns} \\
&= -1 + 2(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots) \\
&= -1 + 2/(1 + e^{-s}) \\
&= (1 - e^{-s})/(1 + e^{-s}) \\
&= (e^{s/2} - e^{-s/2})/(e^{s/2} + e^{-s/2}) \\
s\,F(s) &= \tanh(s/2),
\end{aligned}
$$

because $2\cosh(s/2) = e^{s/2} + e^{-s/2}$ and $2\sinh(s/2) = e^{s/2} - e^{-s/2}$.

35. Let's write $g(t)$ for the on-off function of this problem to distinguish it from the square wave function of Problem 34. Then comparison of Figures 7.2.9 and 7.2.10 makes it clear that $g(t) = \frac{1}{2}\big(1 + f(t)\big)$, so (using the result of Problem 34) we obtain

$$
\begin{aligned}
G(s) &= \frac{1}{2s} + \frac{1}{2}F(s) = \frac{1}{2s} + \frac{1}{2s}\tanh\frac{s}{2} = \frac{1}{2s}\left(1 + \frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}\cdot\frac{e^{-s/2}}{e^{-s/2}}\right) \\
&= \frac{1}{2s}\left(1 + \frac{1 - e^{-s}}{1 + e^{-s}}\right) = \frac{1}{2s}\cdot\frac{2}{1 + e^{-s}} = \frac{1}{s\left(1 + e^{-s}\right)}.
\end{aligned}
$$

36. If $g(t)$ is the triangular wave function of Figure 7.2.11 and $f(t)$ is the square wave function of Problem 34, then $g'(t) = f(t)$. Hence Theorem 1 and the result of Problem 34 yield

$$
\begin{aligned}
\mathcal{L}\{g'(t)\} &= s\,\mathcal{L}\{g(t)\} - g(0), \\
F(s) &= s\,G(s), \qquad \text{(because } g(0) = 0\text{)} \\
\mathcal{L}\{g(t)\} &= s^{-1}F(s) = s^{-2}\tanh(s/2).
\end{aligned}
$$

37. We observe that $f(0) = 0$ and that the sawtooth function has jump -1 at each of the points $t_n = n = 1, 2, 3, \cdots$. Also, $f'(t) \equiv 1$ wherever the derivative is defined. Hence Eq. (21) in this section gives

$$
\frac{1}{s} = s\,F(s) + \sum_{n=1}^{\infty} e^{-ns} = s\,F(s) - 1 + \sum_{n=0}^{\infty} e^{-ns} = s\,F(s) - 1 + \frac{1}{1 - e^{-ns}},
$$

using the geometric series $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$ with $x = e^{-s}$. Solution for $F(s)$ gives

$$F(s) = \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s\left(1-e^{-s}\right)} = \frac{1}{s^2} - \frac{e^{-s}}{s\left(1-e^{-s}\right)}.$$

SECTION 7.3

TRANSLATION AND PARTIAL FRACTIONS

This section is devoted to the computational nuts and bolts of the staple technique for the inversion of Laplace transforms — partial fraction decompositions. If time does not permit going further in this chapter, Sections 7.1–7.3 provide a self-contained introduction to Laplace transforms that suffices for the most common elementary applications.

1. $\mathcal{L}\{t^4\} = \dfrac{24}{s^5}$, so $\mathcal{L}\{t^4 e^{\pi t}\} = \dfrac{24}{(s-\pi)^5}$

2. $\mathcal{L}\{t^{3/2}\} = \dfrac{3\sqrt{\pi}}{4s^{5/2}}$, so $\mathcal{L}\{t^{3/2}\, e^{-4t}\} = \dfrac{3\sqrt{\pi}}{4(s+4)^{5/2}}$.

3. $\mathcal{L}\{\sin 3\pi t\} = \dfrac{3\pi}{s^2+9\pi^2}$, so $\mathcal{L}\{e^{-2t}\sin 3\pi t\} = \dfrac{3\pi}{(s+2)^2+9\pi^2}$.

4.
$$\cos 2\left(t-\frac{\pi}{8}\right) = \cos\left(2t-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\cos 2t + \sin 2t\right)$$
$$\mathcal{L}\left\{\cos 2\left(t-\frac{\pi}{8}\right)\right\} = \frac{1}{\sqrt{2}}\frac{s+2}{s^2+4}$$
$$\mathcal{L}\left\{e^{-t/2}\cos 2\left(t-\frac{\pi}{8}\right)\right\} = \frac{1}{\sqrt{2}}\frac{\left(s+1/2\right)+2}{\left(s+1/2\right)^2+4} = \frac{1}{\sqrt{2}}\frac{2s+5}{4s^2+4s+17}$$

5. $F(s) = \dfrac{3}{2s-4} = \dfrac{3}{2}\cdot\dfrac{1}{s-2}$, so $f(t) = \dfrac{3}{2}e^{2t}$

6. $F(s) = \dfrac{(s+1)-2}{(s+1)^3} = \dfrac{1}{(s+1)^2} - \dfrac{2}{(s+1)^3}$, so $f(t) = te^{-t} - t^2e^{-t} = e^{-t}\left(t-t^2\right)$

7. $F(s) = \dfrac{1}{\left(s+2\right)^2}$, so $f(t) = t\,e^{-2t}$

8. $F(s) = \dfrac{s+2}{\left(s+2\right)^2+1}$, so $f(t) = e^{-2t}\cos t$

9. $F(s) = 3\cdot\dfrac{s-3}{(s-3)^2+16}+\dfrac{7}{2}\cdot\dfrac{4}{(s-3)^2+16}$, so $f(t) = e^{3t}[3\cos 4t + (7/2)\sin 4t]$

10. $$F(s) = \frac{2s-3}{(3s-2)^2+16} = \frac{1}{9}\cdot\frac{2s-3}{(s-2/3)^2+16/9}$$
$$= \frac{2}{9}\cdot\frac{s-2/3}{(s-2/3)^2+(4/3)^2}-\frac{5}{36}\cdot\frac{4/3}{(s-2/3)^2+(4/3)^2}$$
$$f(t) = \frac{1}{36}e^{2t/3}\left(8\cos\frac{4t}{3}-5\sin\frac{4t}{3}\right)$$

11. $F(s) = \dfrac{1}{4}\cdot\dfrac{1}{s-2}-\dfrac{1}{4}\cdot\dfrac{1}{s+2}$, so $f(t) = \dfrac{1}{4}\left(e^{2t}-e^{-2t}\right) = \dfrac{1}{2}\sinh 2t$

12. $F(s) = 2\cdot\dfrac{1}{s}+3\cdot\dfrac{1}{s-3}$, so $f(t) = 2+3e^{3t}$

13. $F(s) = 3\cdot\dfrac{1}{s+2}-5\cdot\dfrac{1}{s+5}$, so $f(t) = 3e^{-2t}-5e^{-5t}$

14. $F(s) = 2\cdot\dfrac{1}{s}-3\cdot\dfrac{1}{s+1}+\dfrac{1}{s-2}$, so $f(t) = 2-3e^{-t}+e^{2t}$

15. $F(s) = \dfrac{1}{25}\left(-1\cdot\dfrac{1}{s}-5\cdot\dfrac{1}{s^2}+\dfrac{1}{s-5}\right)$, so $f(t) = \dfrac{1}{25}\left(-1-5t+e^{5t}\right)$

16. $$F(s) = \frac{1}{(s+3)^2(s-2)^2} = \frac{1}{125}\left(\frac{2}{s+3}+\frac{5}{(s+3)^2}-\frac{2}{s-2}+\frac{5}{(s-2)^2}\right)$$
$$f(t) = \frac{1}{125}\left[e^{-3t}(2+5t)+e^{2t}(-2+5t)\right]$$

17. $$F(s) = \frac{1}{8}\left(\frac{1}{s^2-4}-\frac{1}{s^2+4}\right) = \frac{1}{16}\left(\frac{2}{s^2-4}-\frac{2}{s^2+4}\right)$$
$$f(t) = \frac{1}{16}(\sinh 2t-\sin 2t)$$

18. $$F(s) = \frac{1}{s-4}+\frac{1}{(s-4)^2}+\frac{48}{(s-4)^3}+\frac{64}{(s-4)^4}$$

$$f(t) = e^{4t}\left(1+12t+24t^2+\frac{32}{3}t^3\right)$$

19. $$F(s) = \frac{s^2-2s}{\left(s^2+1\right)\left(s^2+4\right)} = \frac{1}{3}\left(\frac{-2s-1}{s^2+1}+\frac{2s+4}{s^2+4}\right)$$

$$f(t) = \frac{1}{3}\left(-2\cos t-\sin t+2\cos 2t+2\sin 2t\right)$$

20. $$F(s) = \frac{1}{\left(s^2-4\right)^2} = \frac{1}{(s-2)^2(s+2)^2} = \frac{1}{32}\left(\frac{1}{s+2}+\frac{2}{(s+2)^2}-\frac{1}{s-2}+\frac{2}{(s-2)^2}\right)$$

$$f(t) = \frac{1}{32}\left[e^{-2t}(1+2t)+e^{2t}(-1+2t)\right]$$

21. First we need to find A, B, C, D so that

$$\frac{s^2+3}{\left(s^2+2s+2\right)^2} = \frac{As+B}{s^2+2s+2}+\frac{Cs+D}{\left(s^2+2s+2\right)^2}.$$

When we multiply both sides by the quadratic factor s^2+2s+2 and collect coefficients, we get the linear equations

$$\begin{aligned} -2B-D+3 &= 0 \\ -2A-2B-C &= 0 \\ -2A-B+1 &= 0 \\ -A &= 0 \end{aligned}$$

which we solve for $A=0,\ B=1,\ C=-2,\ D=1$. Thus

$$F(s) = \frac{1}{(s+1)^2+1}+\frac{-2s+1}{\left[(s+1)^2+1\right]^2} = \frac{1}{(s+1)^2+1}-2\cdot\frac{s+1}{\left[(s+1)^2+1\right]^2}+3\cdot\frac{1}{\left[(s+1)^2+1\right]^2}.$$

We now use the inverse Laplace transforms given in Eq. (16) and (17) of Section 7.3 — supplying the factor e^{-t} corresponding to the translation $s \to s+1$ — and get

$$f(t) = e^{-t}\left[\sin t-2\cdot\frac{1}{2}t\sin t+3\cdot\frac{1}{2}\left(\sin t-t\cos t\right)\right] = \frac{1}{2}e^{-t}\left(5\sin t-2t\sin t-3t\cos t\right).$$

22. First we need to find A, B, C, D so that

$$\frac{2s^3 - s^2}{\left(4s^2 - 4s + 5\right)^2} = \frac{As + B}{4s^2 - 4s + 5} + \frac{Cs + D}{\left(4s^2 - 4s + 5\right)^2}.$$

When we multiply each side by $(4s^2 - 4s + 5)^2$ we get the identity

$$2s^3 - s^2 = (As + B)(4s^2 - 4s + 5) + Cs + D.$$

When we substitute the root $s = 1/2 + i$ of the quadratic into this identity, we find that $C = -3/2$ and $D = -5/4$. When we first differentiate each side of the identity and then substitute the root, we find that $A = 1/2$ and $B = 1/4$. Writing

$$4s^2 - 4s + 5 = 4[(s - 1/2)^2 + 1],$$

it follows that

$$F(s) = \frac{1}{8} \cdot \frac{\left(s - \frac{1}{2}\right) + 1}{\left(s - \frac{1}{2}\right)^2 + 1} - \frac{1}{32} \cdot \frac{3\left(s - \frac{1}{2}\right) + 4}{\left[\left(s - \frac{1}{2}\right)^2 + 1\right]^2}.$$

Finally the results

$$\mathcal{L}^{-1}\{2s/(s^2 + 1)^2\} = t \sin t$$

$$\mathcal{L}^{-1}\{2/(s^2 + 1)^2\} = \sin t - t \cos t$$

of Eqs. (16) and (17) in Section 7.3, together with the translation theorem, yield

$$f(t) = e^{t/2}\left[\frac{1}{8} \cdot (\cos t + \sin t) - \frac{3}{32} \cdot \frac{1}{2} t \sin t - \frac{4}{32} \cdot \frac{1}{2}(\sin t - t \cos t)\right]$$

$$= \frac{1}{64} e^{t/2}\left[(8 + 4t)\cos t + (4 - 3t)\sin t\right].$$

23. $$\frac{s^3}{s^4 + 4a^4} = \frac{1}{2}\left(\frac{s - a}{s^2 - 2as + 2a^2} + \frac{s + a}{s^2 + 2as + 2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4 + 4a^4}\right\} = \frac{1}{2}\left(e^{at} + e^{-at}\right)\cos at = \cosh at \cos at.$$

24. $$\frac{s}{s^4 + 4a^4} = \frac{1}{4a^2}\left(\frac{a}{s^2 - 2as + 2a^2} - \frac{a}{s^2 + 2as + 2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \frac{1}{4a^2}\left(e^{at}-e^{-at}\right)\sin at = \frac{1}{2a^2}\sinh at\sin at.$$

25.
$$\begin{aligned}\frac{s}{s^4+4a^4} &= \frac{1}{4a}\left(\frac{s}{s^2-2as+2a^2}-\frac{s}{s^2+2as+2a^2}\right)\\ &= \frac{1}{4a}\left(\frac{s-a}{s^2-2as+2a^2}+\frac{a}{s^2-2as+2a^2}-\frac{s+a}{s^2+2as+2a^2}+\frac{a}{s^2+2as+2a^2}\right),\end{aligned}$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} &= \frac{1}{4a}\left[e^{at}(\cos at+\sin at)-e^{-at}\left(\cos at-\sin at\right)\right]\\ &= \frac{1}{2a}\left[\frac{1}{2}\left(e^{at}+e^{-at}\right)\sin at+\frac{1}{2}\left(e^{at}-e^{-at}\right)\cos at\right]\\ &= \frac{1}{2a}\left(\cosh at\sin at+\sinh at\cos at\right).\end{aligned}$$

26.
$$\begin{aligned}\frac{1}{s^4+4a^4} &= \frac{1}{8a^3}\left(\frac{-s+2a}{s^2-2as+2a^2}+\frac{s+2a}{s^2+2as+2a^2}\right)\\ &= \frac{1}{8a^3}\left(-\frac{s-a}{s^2-2as+2a^2}+\frac{a}{s^2-2as+2a^2}+\frac{s+a}{s^2+2as+2a^2}+\frac{a}{s^2+2as+2a^2}\right),\end{aligned}$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} &= \frac{1}{8a^3}\left[e^{at}(-\cos at+\sin at)+e^{-at}\left(\cos at+\sin at\right)\right]\\ &= \frac{1}{4a^3}\left[\frac{1}{2}\left(e^{at}+e^{-at}\right)\sin at-\frac{1}{2}\left(e^{at}-e^{-at}\right)\cos at\right]\\ &= \frac{1}{4a^3}\left(\cosh at\sin at-\sinh at\cos at\right).\end{aligned}$$

In Problems 27–40 we give first the transformed equation, then the Laplace transform $X(s)$ of the solution, and finally the desired solution $x(t)$.

27. $[s^2X(s) - 2s - 3] + 6[sX(s) - 2] + 25X(s) = 0$

$$X(s) = \frac{2s+15}{s^2+6s+25} = 2\cdot\frac{s+3}{(s+3)^2+16}+\frac{9}{4}\cdot\frac{4}{(s+3)^2+16}$$

$x(t) = e^{-3t}[2\cos 4t + (9/4)\sin 4t]$

28. $s^2X(s)-6sX(s)+8X(s) = \dfrac{2}{s}$

$$X(s) = \frac{2}{s(s^2-6s+8)} = \frac{1}{4}\left(\frac{1}{s}+\frac{1}{s-4}-\frac{2}{s-2}\right)$$

$$x(t) = \frac{1}{4}\left(1+e^{4t}-2e^{2t}\right)$$

29. $s^2X(s)-4X(s) = \dfrac{3}{s^2}$

$$X(s) = \frac{3}{s^2\left(s^2-4\right)} = \frac{3}{4}\left(\frac{1}{s^2-4}-\frac{1}{s^2}\right)$$

$$x(t) = \frac{3}{8}\sinh 2t-\frac{3}{4}t = \frac{3}{8}\left(\sinh 2t-2t\right)$$

30. $s^2X(s)+4sX(s)+8X(s) = \dfrac{1}{s+1}$

$$X(s) = \frac{1}{(s+1)\left(s^2+4s+8\right)} = \frac{1}{5}\left(\frac{1}{s+1}-\frac{s+3}{s^2+4s+8}\right)$$

$$= \frac{1}{5}\left(\frac{1}{s+1}-\frac{s+2}{(s+2)^2+4}-\frac{1}{2}\cdot\frac{2}{(s+2)^2+4}\right)$$

$$x(t) = \frac{1}{10}\left[2e^{-t}-e^{-2t}(2\cos 2t+\sin 2t)\right]$$

31. $[s^3X(s) - s - 1]+[s^2X(s) - 1] - 6[sX(s)] = 0$

$$X(s) = \frac{s+2}{s^3+s^2-6s} = \frac{1}{15}\left(-\frac{5}{s}-\frac{1}{s+3}+\frac{6}{s-2}\right)$$

$$x(t) = \frac{1}{15}\left(-5-e^{-3t}+6e^{2t}\right)$$

32. $\left[s^4X(s)-s^3\right]-X(s) = 0$

$$X(s) = \frac{s^3}{s^4-1} = \frac{1}{2}\left(\frac{s}{s^2+1}+\frac{s}{s^2-1}\right)$$

$$x(t) = \frac{1}{2}\left(\cos t+\cosh t\right)$$

33. $[s^4X(s) - 1] + X(s) = 0$

$$X(s) = \frac{1}{s^4+1}$$

It therefore follows from Problem 26 with $a = \sqrt[4]{1/4} = 1/\sqrt{2}$ that

$$x(t) = \frac{1}{\sqrt{2}}\left(\cosh\frac{t}{\sqrt{2}}\sin\frac{t}{\sqrt{2}} - \sinh\frac{t}{\sqrt{2}}\cos\frac{t}{\sqrt{2}}\right).$$

34. $[s^4X(s) - 2s^2 + 13] + 13[s^2X(s) - 2] + 36\,X(s) = 0$

$$X(s) = \frac{2s^2+13}{s^4+13s^2+36} = \frac{1}{s^2+4} + \frac{1}{s^2+9}$$

$$x(t) = \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t$$

35. $\left[s^4X(s) - 1\right] + 8s^2X(s) + 16X(s) = 0$

$$X(s) = \frac{1}{s^4+8s^2+16} = \frac{1}{\left(s^2+4\right)^2}$$

$$x(t) = \frac{1}{16}\left(\sin 2t - 2t\cos 2t\right) \qquad \text{(by Eq. (17) in Section 7.3)}$$

36. $s^4X(s) + 2s^2X(s) + X(s) = \dfrac{1}{s-2}$

$$X(s) = \frac{1}{(s-2)\left(s^4+2s^2+1\right)} = \frac{1}{25}\left(\frac{1}{s-2} - \frac{s+2}{s^2+1} - \frac{5(s+2)}{\left(s^2+1\right)^2}\right)$$

$$x(t) = \frac{1}{25}\left(e^{-2t} - \cos t - 2\sin t - 5\cdot\frac{1}{2}t\sin t - 10\cdot\frac{1}{2}\left(\sin t - t\cos t\right)\right)$$

$$= \frac{1}{50}\left[2e^{2t} + (10t-2)\cos t - (5t+14)\sin t\right]$$

37. $\left[s^2X(s) - 2\right] + 4sX(s) + 13X(s) = \dfrac{1}{\left(s+1\right)^2}$

$$X(s) = \frac{2+1/(s+1)^2}{s^2+4s+13} = \frac{2s^2+4s+3}{(s+1)^2\left(s^2+4s+13\right)}$$

$$= \frac{1}{50}\left[-\frac{1}{s+1}+\frac{5}{(s+1)^2}+\frac{s+98}{(s+2)^2+9}\right]$$

$$= \frac{1}{50}\left[-\frac{1}{s+1}+\frac{5}{(s+1)^2}+\frac{s+2}{(s+2)^2+9}+32\cdot\frac{3}{(s+2)^2+9}\right]$$

$$x(t) = \frac{1}{50}\left[(-1+5t)e^{-t}+e^{-2t}(\cos 3t+32\sin 3t)\right]$$

38. $\left[s^2X(s)-s+1\right]+6\left[sX(s)-1\right]+18X(s) = \frac{s}{s^2+4}$

$$X(s) = \frac{s+5}{s^2+6s+18}+\frac{s}{\left(s^2+4\right)\left(s^2+6s+18\right)}$$

$$= \frac{s+5}{s^2+6s+18}+\frac{1}{170}\left(\frac{7s+12}{s^2+4}-\frac{7s+54}{s^2+6s+18}\right)$$

$$= \frac{1}{170}\left(\frac{7s+12}{s^2+4}+\frac{163s+796}{s^2+6s+18}\right)$$

$$X(s) = \frac{1}{170}\left(\frac{7s+12}{s^2+4}+\frac{163(s+3)}{(s+3)^2+9}+\frac{307}{(s+3)^2+9}\right)$$

$$x(t) = \frac{1}{170}\left(7\cos 2t+6\sin 2t\right)+\frac{1}{510}e^{-3t}\left(489\cos 3t+307\sin 3t\right)$$

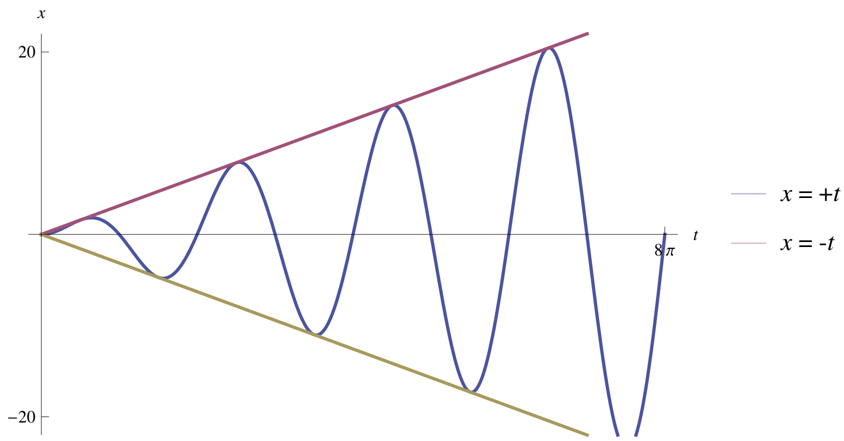
39. $x''+9x = 6\cos 3t, \quad x(0)=x'(0)=0$

$$s^2X(s)+9X(s) = \frac{6s}{s^2+9}$$

$$X(s) = \frac{6s}{\left(s^2+9\right)^2}$$

$$x(t) = 6\cdot\frac{1}{2\cdot 3}t\sin 3t = t\sin 3t \qquad \text{(by Eq. (16) in Section 7.3)}$$

The graph of this resonance is shown in the figure at the top of the next page.



40. $x'' + 0.4x' + 9.04x = x'' + \frac{2}{5}x' + \frac{226}{25} = 6e^{-t/5}\cos 3t$

$$\left(s^2 + \frac{2}{5}s + \frac{226}{25}\right)X(s) = \frac{6(s+1/5)}{(s+1/5)^2+9}$$

$$X(s) = \frac{6(s+1/5)}{\left[(s+1/5)^2+9\right]^2}$$

$$x(t) = t\,e^{-t/5}\sin 3t \qquad \text{(by Eq. (16) in Section 7.3)}$$

SECTION 7.4

DERIVATIVES, INTEGRALS, AND PRODUCTS OF TRANSFORMS

This section completes the presentation of the standard "operational properties" of Laplace transforms, the most important one here being the convolution property $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$, where the **convolution** $f*g$ is defined by

$$f*g(t) = \int_0^t f(x)g(t-x)\,dx.$$

Here we use x rather than τ as the variable of integration; compare with Eq. (3) in Section 7.4 of the textbook.

1. With $f(t)=t$ and $g(t)=1$ we calculate

$$t*1 = \int_0^t x\cdot 1\,dx = \left[\frac{1}{2}x^2\right]_{x=0}^{x=t} = \frac{1}{2}t^2.$$

2. With $f(t)=t$ and $g(t)=e^{at}$ we calculate

$$\begin{aligned} t*e^{at} &= \int_0^t x\cdot e^{a(t-x)}\,dx = e^{at}\int_0^t x\cdot e^{-ax}\,dx \\ &= e^{at}\int_{-}^{-}\left(-\frac{u}{a}\right)e^{u}\left(-\frac{du}{a}\right) = \frac{e^{at}}{a^2}\int_{-}^{-} ue^{u}\,du \qquad (\text{with } u=-ax) \\ &= \frac{e^{at}}{a^2}\Big[(u-1)e^{u}\Big]_{-}^{-} \qquad (\text{integral formula \#46 inside back cover}) \\ &= \frac{e^{at}}{a^2}\Big[(-ax-1)e^{-ax}\Big]_{x=0}^{x=t} = \frac{e^{at}}{a^2}\Big[(-at-1)e^{-at}+1\Big] \\ t*e^{at} &= \frac{1}{a^2}\left(e^{at}-at-1\right). \end{aligned}$$

3. To compute $(\sin t)*(\sin t) = \int_0^t \sin x\sin(t-x)\,dx$, we first apply the identity $\sin A\sin B = [\cos(A - B) - \cos(A + B)]/2$. This gives

$$\begin{aligned} (\sin t)*(\sin t) &= \int_0^t \sin x\sin(t-x)\,dx \\ &= \frac{1}{2}\int_0^t\big[\cos(2x-t)-\cos t\big]dx \\ &= \frac{1}{2}\left[\frac{1}{2}\sin(2x-t)-x\cos t\right]_{x=0}^{x=t} \\ (\sin t)*(\sin t) &= \frac{1}{2}\left(\sin t - t\cos t\right). \end{aligned}$$

4. To compute $t^2*\cos t = \int_0^t x^2\cos(t-x)\,dx$, we first substitute

$$\cos(t - x) = \cos t\cos x + \sin t\sin x,$$

and then use the integral formulas

$$\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C$$

$$\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

from #40 and #41 inside the back cover of the textbook. This gives

$$\begin{aligned}
t^2 * \cos t &= \int_0^t x^2(\cos t \cos x + \sin t \sin x)\,dx \\
&= (\cos t)\int_0^t x^2 \cos x\,dx + (\sin t)\int_0^t x^2 \sin x\,dx \\
&= (\cos t)\Big[x^2 \sin x + 2x\cos x - 2\sin x\Big]_{x=0}^{x=t} \\
&\qquad + (\sin t)\Big[-x^2 \cos x + 2x \sin x + 2\cos x\Big]_{x=0}^{x=t} \\
t^2 * \cos t &= 2(t - \sin t).
\end{aligned}$$

5. $$e^{at} * e^{at} = \int_0^t e^{ax}e^{a(t-x)}dx = \int_0^t e^{at}dx = e^{at}\big[x\big]_{x=0}^{x=t} = t\,e^{at}$$

6. $$\begin{aligned}
e^{at} * e^{bt} &= \int_0^t e^{ax}e^{b(t-x)}dx = e^{bt}\int_0^t e^{(a-b)x}dx \\
&= e^{bt}\left[\frac{e^{(a-b)x}}{a-b}\right]_{x=0}^{x=t} = \frac{e^{bt}\left(e^{(a-b)t}-1\right)}{a-b} = \frac{e^{at}-e^{bt}}{a-b}
\end{aligned}$$

7. $$f(t) = 1 * e^{3t} = e^{3t} * 1 = \int_0^t e^{3x}\cdot 1\,dx = \frac{1}{3}\left(e^{3t}-1\right)$$

8. $$f(t) = 1 * \frac{1}{2}\sin 2t = \int_0^t \frac{1}{2}\sin 2x\,dx = \frac{1}{4}(1-\cos 2t)$$

9. $$\begin{aligned}
f(t) &= \frac{1}{9}\sin 3t * \sin 3t = \frac{1}{9}\int_0^t \sin 3x \sin 3(t-x)\,dx \\
&= \frac{1}{9}\int_0^t \sin 3x\big[\sin 3t \cos 3x - \cos 3t \sin 3x\big]dx \\
&= \frac{1}{9}\sin 3t\int_0^t \sin 3x \cos 3x\,dx - \frac{1}{9}\cos 3t\int_0^t \sin^2 3x\,dx \\
&= \frac{1}{9}\sin 3t\left[\frac{1}{6}\sin^2 3x\right]_{x=0}^{x=t} - \frac{1}{9}\cos 3t\left[\frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right)\right]_{x=0}^{x=t} \\
f(t) &= \frac{1}{54}\left(\sin 3t - 3t\cos 3t\right)
\end{aligned}$$

10. $$f(t) = t * (\sin kt)/k = \frac{1}{k}\int_0^t \sin kx \cdot (t-x)\,dx$$

$$= \frac{t}{k}\int_0^t \sin kx\,dx - \frac{1}{k}\int_0^t x\sin kx\,dx = \frac{kt-\sin kt}{k^3}$$

11. $f(t) = \cos 2t * \cos 2t = \int_0^t \cos 2x\,\cos 2(t-x)\,dx$

$$= \int_0^t \cos 2x\left(\cos 2t\cos 2x + \sin 2t\sin 2x\right)dx$$

$$= (\cos 2t)\int_0^t \cos^2 2x\,dx + (\sin 2t)\int_0^t \cos 2x\sin 2x\,dx$$

$$= (\cos 2t)\left[\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]_{x=0}^{x=t} + (\sin 2t)\left[\frac{1}{4}\sin^2 2x\right]_{x=0}^{x=t}$$

$$f(t) = \frac{1}{4}\left(\sin 2t + 2t\cos 2t\right)$$

12. $f(t) = (e^{-2t}\sin t)*(1) = \int_0^t e^{-2x}\sin x\,dx = \frac{1}{5}\left[1-e^{-2t}(\cos t+2\sin t)\right]$

13. $f(t) = e^{3t} * \cos t = \int_0^t (\cos x)e^{3(t-x)}dx$

$$= e^{3t}\int_0^t e^{-3x}\cos x\,dx$$

$$= e^{3t}\left[\frac{e^{-3x}}{10}\left(-3\cos x+\sin x\right)\right]_{x=0}^{x=t} \qquad \text{(by integral formula \#50)}$$

$$f(t) = \frac{1}{10}\left(3e^{3t} - 3\cos t + \sin t\right)$$

14. $f(t) = \cos 2t * \sin t = \int_0^t \cos 2x\sin(t-x)\,dx$

$$= \int_0^t \cos 2x\,(\sin t\cos x - \cos t\sin x)\,dx$$

$$= (\sin t)\int_0^t \cos 2x\cos x\,dx - (\cos t)\int_0^t \cos 2x\sin x\,dx$$

$$= \frac{1}{2}(\sin t)\int_0^t \left(\cos 3x+\cos x\right)dx - \frac{1}{2}(\cos t)\int_0^t \left(\sin 3x - \sin x\right)dx$$

$$f(t) = \frac{1}{3}\left(\cos t - \cos 2t\right)$$

15. $\mathcal{L}\{t\sin 3t\} = -\frac{d}{ds}\left(\mathcal{L}\{\sin 3t\}\right) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{\left(s^2+9\right)^2}$

16. $\mathcal{L}\{t^2\cos 2t\} = \dfrac{d^2}{ds^2}\left(\mathcal{L}\{\cos 2t\}\right) = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+4}\right) = \dfrac{2s(s^2-12)}{\left(s^2+4\right)^3}$

17. $\mathcal{L}\{e^{2t}\cos 3t\} = (s-2)/(s^2-4s+13)$

$\mathcal{L}\{te^{2t}\cos 3t\} = -(d/ds)[(s-2)/(s^2-4s+13)] = (s^2-4s-5)/(s^2-4s+13)^2$

18. $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\{(1-\cos 2t)/2\} = 2/s(s^2+4)$

$\mathcal{L}\{e^{-t}\sin^2 t\} = 2/[(s+1)(s^2+2s+5)]$

$\mathcal{L}\{te^{-t}\sin^2 t\} = -(d/ds)[2/((s+1)(s^2+2s+5))]$

$= 2(3s^2+6s+7)/[(s+1)^2(s^2+2s+5)^2]$

19. $\mathcal{L}\left\{\dfrac{\sin t}{t}\right\} = \displaystyle\int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2}-\tan^{-1}s = \tan^{-1}\left(\frac{1}{s}\right)$

20. $\mathcal{L}\{1-\cos 2t\} = \dfrac{1}{s}-\dfrac{s}{s^2+4}$, so

$$\mathcal{L}\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+4}\right)ds = \left[\ln\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty = \ln\left(\frac{\sqrt{s^2+4}}{s}\right)$$

21. $\mathcal{L}\left\{e^{3t}-1\right\} = \dfrac{1}{s-3}-\dfrac{1}{s}$, so

$$\mathcal{L}\left\{\frac{e^{3t}-1}{t}\right\} = \int_s^\infty\left(\frac{1}{s-3}-\frac{1}{s}\right)ds = \left[\ln\left(\frac{s-3}{s}\right)\right]_s^\infty = \ln\left(\frac{s}{s-3}\right)$$

22. $\mathcal{L}\left\{e^{t}-e^{-t}\right\} = \dfrac{1}{s-1}-\dfrac{1}{s+1} = \dfrac{2}{s^2-1}$, so

$$\mathcal{L}\left\{\frac{e^{t}-e^{-t}}{t}\right\} = \int_s^\infty\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds = \left[\ln\left(\frac{s-1}{s+1}\right)\right]_s^\infty = \ln\left(\frac{s+1}{s-1}\right)$$

23. $f(t) = -\dfrac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\dfrac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}-\dfrac{1}{s+2}\right\} = -\dfrac{1}{t}\left(e^{2t}-e^{-2t}\right) = -\dfrac{2\sinh 2t}{t}$

24. $f(t) = -\dfrac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\dfrac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2+1}-\dfrac{2s}{s^2+4}\right\} = \dfrac{2}{t}\left(\cos 2t-\cos t\right)$

25. $$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s+2} - \frac{1}{s-3}\right\} = \frac{1}{t}\left(e^{-2t} + e^{3t} - 2\cos t\right)$$

26. $$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{3}{(s+2)^2+9}\right\} = \frac{e^{-2t}\sin 3t}{t}$$

27. $$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{-2/s^3}{1+1/s^2}\right\}$$

$$= \frac{2}{t}\mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\} = \frac{2}{t}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = \frac{2}{t}(1-\cos t)$$

28. An empirical approach works best with this one. We can construct transforms with powers of $(s^2 + 1)$ in their denominators by differentiating the transforms of $\sin t$ and $\cos t$. Thus,

$$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}$$

$$\mathcal{L}\{t^2\cos t\} = -\frac{d}{ds}\left(\frac{s^2-1}{(s^2+1)^2}\right) = \frac{2s^3-6s}{(s^2+1)^3}.$$

From the first and last of these formulas it follows readily that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^3}\right\} = \frac{1}{8}\left(t\sin t - t^2\cos t\right).$$

Alternatively, one could work out the repeated convolution

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^3}\right\} = (\cos t) * (\sin t * \sin t).$$

29. $-[s^2X(s) - x'(0)]' - [s\,X(s)]' - 2[s\,X(s)] + X(s) = 0$

$s(s + 1)X'(s) + 4s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s+1)^4} \quad \text{with } A \neq 0$$

$$x(t) = Ct^3 e^{-t} \quad \text{with } C \neq 0$$

30. $-[s^2 X(s) - x'(0)]' - 3[s\,X(s)]' - [s\,X(s)] + 3X(s) = 0$

$-(s^2 + 3s)X'(s) - 3s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s+3)^3} \quad \text{with } A \neq 0$$

$$x(t) = Ct^2 e^{-3t} \text{ with } C \neq 0$$

31. $-[s^2 X(s) - x'(0)]' + 4[s\,X(s)]' - [s\,X(s)] - 4[X(s)]' + 2X(s) = 0$

$(s^2 - 4s + 4)X'(s) + (3s - 6)X(s) = 0$ (separable)

$(s - 2)X'(s) + 3X(s) = 0$

$$X(s) = \frac{A}{(s-2)^3} \quad \text{with } A \neq 0$$

$$x(t) = Ct^2 e^{2t} \quad \text{with } C \neq 0$$

32. $-[s^2 X(s) - x'(0)]' - 2[s\,X(s)]' - 2[s\,X(s)] - 2X(s) = 0$

$-(s^2 + 2s)X'(\text{s}) - (4s + 4)X(s) = 0$ (separable)

$$X(s) = \frac{A}{s^2(s+2)^2} = C\left[\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+2} - \frac{1}{(s+2)^2}\right]$$

$$x(t) = C(1 - t - e^{-2t} - te^{-2t}) \quad \text{with } C = -A/4 \neq 0$$

33. $-[s^2 X(s) - x(0)]' - 2[s\,X(s)] - [X(s)]' = 0$

$(s^2 + 1)X'(s) + 4s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s^2+1)^2} \quad \text{with } A \neq 0$$

$$x(t) = C(\sin t - t\cos t) \quad \text{with } C \neq 0$$

34. $-(s^2 + 4s + 13)X'(s) - (4s + 8)X(s) = 0$

$$X(s) = \frac{C}{(s^2+4s+13)^2} = \frac{C}{\left[(s+2)^2+9\right]^2}$$

It now follows from Problem 31 in Section 7.2 that

$$x(t) = Ae^{-2t}(\sin 3t - 3t\cos 3t) \text{ with } A \neq 0.$$

35. $$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\} = e^t * \frac{1}{\sqrt{\pi t}} = \int_0^t \frac{1}{\sqrt{\pi x}}\cdot e^{t-x}\,dx$$

$$= \frac{e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} \frac{1}{u}\cdot e^{-u^2}\cdot 2u\,du = \frac{2e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du = e^t \operatorname{erf}\left(\sqrt{t}\right)$$

36. $s^2X(s) + 4X(s) = F(s)$

$$X(s) = \frac{1}{2}F(s)\cdot\frac{2}{s^2+4}$$

$$x(t) = \frac{1}{2}f(t)*\sin 2t = \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\,d\tau$$

37. $s^2X(s)+2sX(s)+X(s) = F(s)$

$$X(s) = F(s)\cdot\frac{1}{(s+1)^2}$$

$$x(t) = te^{-t}*f(t) = \int_0^t \tau e^{-\tau}f(t-\tau)\,d\tau$$

38. $s^2X(s)+4sX(s)+13X(s) = F(s)$

$$X(s) = \frac{F(s)}{s^2+4s+13} = \frac{1}{3}F(s)\cdot\frac{3}{(s+2)^2+9}$$

$$x(t) = \frac{1}{3}f(t)*e^{-2t}\sin 3t = \frac{1}{3}\int_0^t e^{-2\tau}f\left(t-\tau\right)\sin 3\tau\,d\tau$$

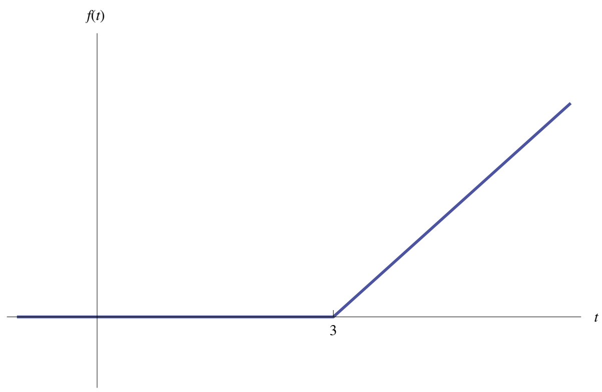
SECTION 7.5

PERIODIC AND PIECEWISE CONTINUOUS INPUT FUNCTIONS

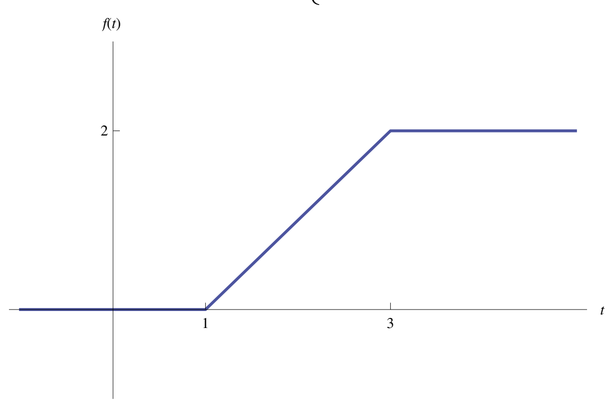
In Problems 1 through 10, we first derive the inverse Laplace transform $f(t)$ of $F(s)$ and then show the graph of $f(t)$.

1. $F(s) = e^{-3s}\mathcal{L}\{t\}$ so Eq. (3b) in Theorem 1 gives

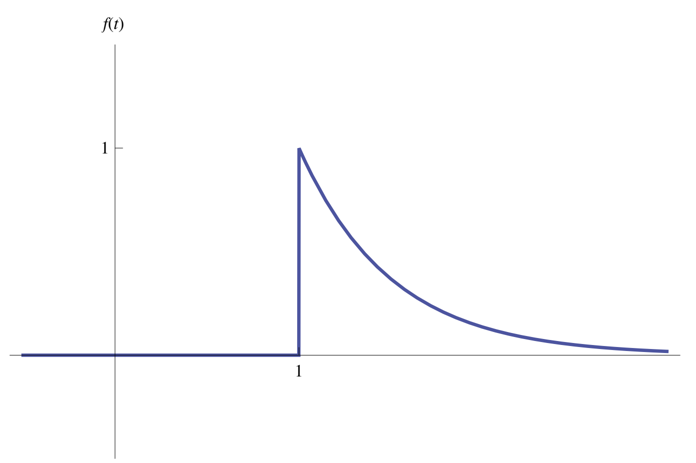
$$f(t) = u(t-3)\cdot(t-3) = \begin{cases} 0 & \text{if } t<3, \\ t-3 & \text{if } t\geq 3. \end{cases}$$



2. $$f(t) = (t-1)u(t-1)-(t-3)u(t-3) = \begin{cases} 0 & \text{if } t<1, \\ t-1 & \text{if } 1\leq t<3, \\ 2 & \text{if } t\geq 3. \end{cases}$$

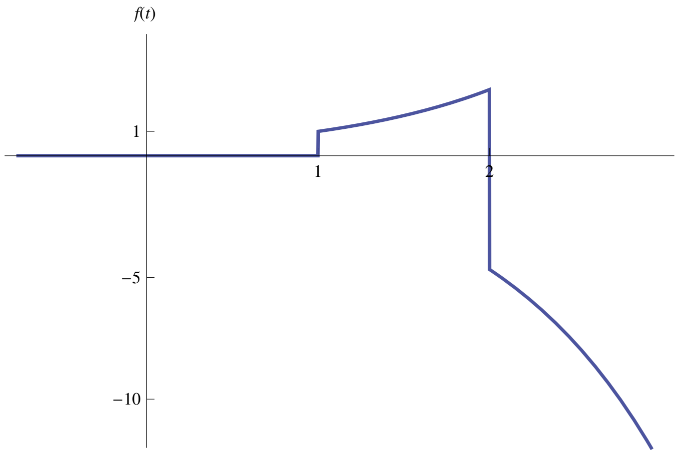


3. $F(s) = e^{-s}\mathcal{L}\{e^{-2t}\}$ so $f(t) = u(t-1)\cdot e^{-2(t-1)} = \begin{cases} 0 & \text{if } t<1, \\ e^{-2(t-1)} & \text{if } t\geq 1. \end{cases}$



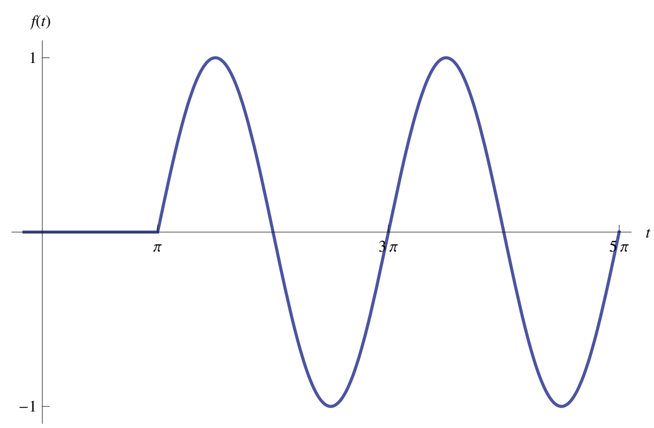
4. $F(s) = e^{-s}\mathcal{L}\{t\} - e^2 e^{-2s}\mathcal{L}\{t\}$ so

$$f(t) = e^{t-1}u(t-1) - e^2 e^{t-2}u(t-2) = \begin{cases} 0 & \text{if } t<1, \\ e^{t-1} & \text{if } 1 \le t < 2, \\ e^{t-1} - e^t & \text{if } t \ge 2. \end{cases}$$



5. $F(s) = e^{-\pi s}\mathcal{L}\{\sin t\}$ so

$$f(t) = u(t-\pi)\cdot\sin(t-\pi) = -u(t-\pi)\sin t = \begin{cases} 0 & \text{if } t<\pi, \\ -\sin t & \text{if } t \ge \pi. \end{cases}$$



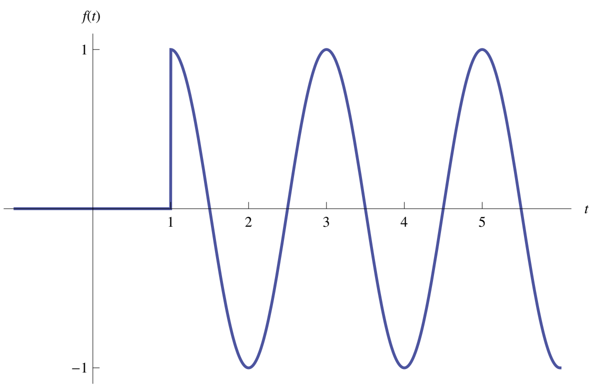
6. $F(s) = e^{-s}\mathcal{L}\{\cos \pi t\}$ so

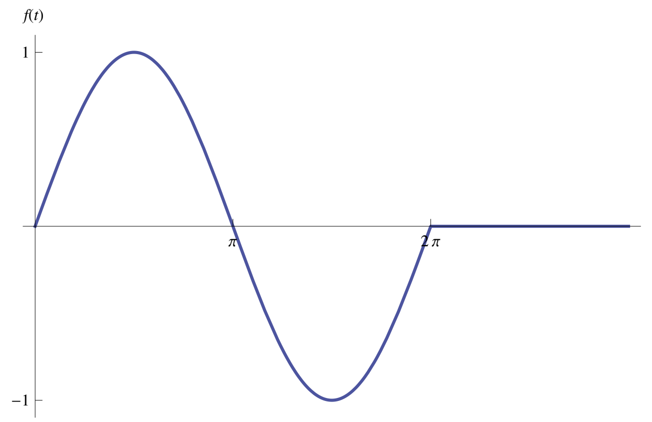
$$f(t) = u(t-1)\cdot\cos\pi(t-1) = -u(t-1)\cos\pi t = \begin{cases} 0 & \text{if } t<1, \\ -\cos\pi t & \text{if } t \ge 1. \end{cases}$$

7. $F(s) = \mathcal{L}\{\sin t\} - e^{-2\pi s}\mathcal{L}\{\sin t\}$ so

$$f(t) = \sin t - u(t-2\pi)\sin(t-2\pi) = [1-u(t-2\pi)]\sin t = \begin{cases} \sin t & \text{if } t<2\pi, \\ 0 & \text{if } t \ge 2\pi. \end{cases}$$

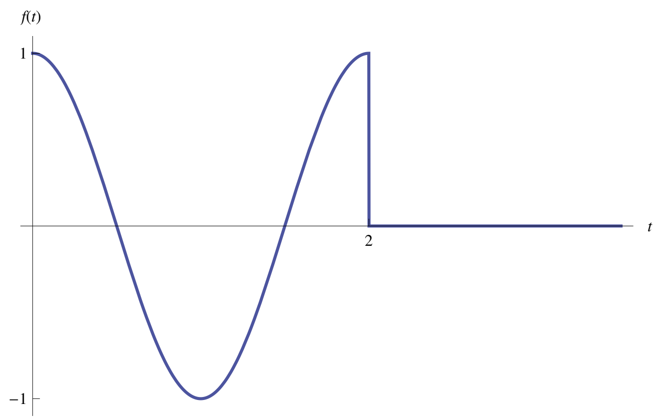
The left-hand figure below is the graph for Problem 6 on the preceding page, and the right-hand figure is the graph for Problem 7.





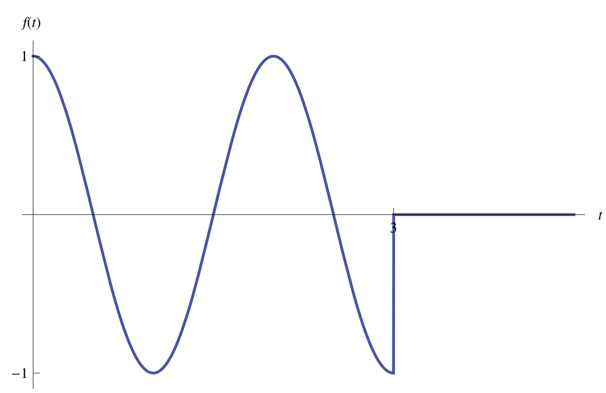
8. $F(s) = \mathcal{L}\{\cos \pi t\} - e^{-2s}\mathcal{L}\{\cos \pi t\}$ so

$$f(t) = \cos \pi t - u(t-2)\cos \pi(t-2) = [1-u(t-2)]\cos \pi t = \begin{cases} \cos \pi t & \text{if } t < 2, \\ 0 & \text{if } t \geq 2. \end{cases}$$



9. $F(s) = \mathcal{L}\{\cos \pi t\} + e^{-3s}\mathcal{L}\{\cos \pi t\}$ so

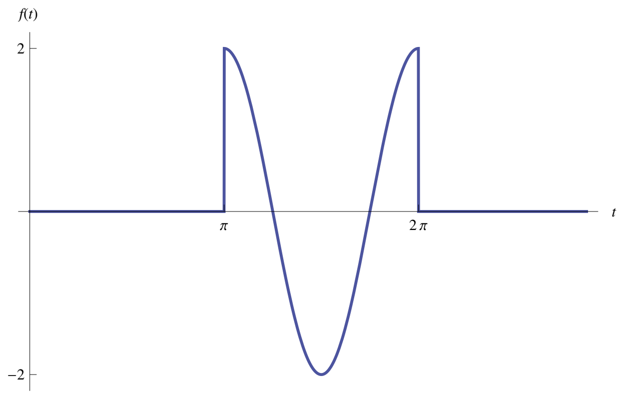
$$f(t) = \cos \pi t + u(t-3)\cos \pi(t-3) = [1-u(t-3)]\cos \pi t = \begin{cases} \cos \pi t & \text{if } t < 3, \\ 0 & \text{if } t \geq 3. \end{cases}$$



10. $F(s) = e^{-\pi s}\mathcal{L}\{2\cos 2t\} + e^{-2\pi s}\mathcal{L}\{2\cos 2t\}$ so

$$f(t) = 2u(t-\pi)\cos 2(t-\pi) - 2u(t-2\pi)\cos 2(t-2\pi)$$

$$= 2\left[u(t-\pi) - u(t-2\pi)\right]\cos 2t = \begin{cases} 0 & \text{if } t < \pi \text{ or } t \ge 2\pi, \\ 2\cos 2t & \text{if } \pi \le t < 2\pi. \end{cases}$$



11. $f(t) = 2 - u(t-3)\cdot 2$ so $F(s) = \dfrac{2}{s} - e^{-3s}\dfrac{2}{s} = \dfrac{2}{s}\left(1 - e^{-3s}\right).$

12. $f(t) = u(t-1) - u(t-4)$ so $F(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-3s}}{s} = \dfrac{1}{s}\left(e^{-s} - e^{-3s}\right).$

13. $f(t) = [1 - u(t-2\pi)]\sin t = \sin t - u(t-2\pi)\sin(t-2\pi)$ so

$$F(s) = \frac{1}{s^2+1} - e^{-2\pi s}\cdot\frac{1}{s^2+1} = \frac{1-e^{-2\pi s}}{s^2+1}.$$

14. $f(t) = [1 - u(t-2)]\cos \pi t = \cos \pi t - u(t-2)\cos \pi(t-2)$ so

$$F(s) = \frac{s}{s^2+\pi^2} - e^{-2s}\cdot\frac{s}{s^2+\pi^2} = \frac{s\left(1-e^{-2s}\right)}{s^2+\pi^2}.$$

15. $f(t) = [1 - u(t-3\pi)]\sin t = \sin t + u(t-3\pi)\sin(t-3\pi)$ so

$$F(s) = \frac{1}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1} = \frac{1+e^{-3\pi s}}{s^2+1}.$$

16. $f(t) = [u(t-\pi) - u(t-2\pi)]\sin 2t = u(t-\pi)\sin 2(t-\pi) - u(t-2\pi)\sin 2(t-2\pi)$ so

$$F(s) = \left(e^{-\pi s} - e^{-2\pi s}\right)\cdot\frac{2}{s^2+4} = \frac{2\left(e^{-\pi s} - e^{-2\pi s}\right)}{s^2+4}.$$

17. $f(t) = [u(t-2)-u(t-3)]\sin \pi t = u(t-2)\sin \pi(t-2) + u(t-3)\sin \pi(t-3)$ so

$$F(s) = \left(e^{-2s}+e^{-3s}\right)\cdot\frac{\pi}{s^2+\pi^2} = \frac{\pi\left(e^{-2s}+e^{-3s}\right)}{s^2+\pi^2}.$$

18. $f(t) = [u(t-3)-u(t-5)]\cos\frac{\pi t}{2} = u(t-3)\sin\frac{\pi}{2}(t-3) + u(t-5)\sin\frac{\pi}{2}(t-5)$ so

$$F(s) = \left(e^{-3s}+e^{-5s}\right)\cdot\frac{\pi/2}{s^2+\pi^2/4} = \frac{2\pi\left(e^{-3s}+e^{-5s}\right)}{4s^2+\pi^2}.$$

19. If $g(t) = t+1$ then $f(t) = u(t-1)\cdot t = u(t-1)\cdot g(t-1)$ so

$$F(s) = e^{-s}G(s) = e^{-s}L\{t+1\} = e^{-s}\cdot\left(\frac{1}{s^2}+\frac{1}{s}\right) = \frac{e^{-s}(s+1)}{s^2}.$$

20. If $g(t)=t+1$ then $f(t) = [1-u(t-1)]t+u(t-1) = t-u(t-1)g(t-1)+u(t-1)$ so

$$F(s) = \frac{1}{s^2}-e^{-s}\cdot G(s)+\frac{e^{-s}}{s} = \frac{1}{s^2}-e^{-s}\cdot\left(\frac{1}{s^2}+\frac{1}{s}\right)+\frac{e^{-s}}{s} = \frac{1-e^{-s}}{s^2}.$$

21. If $g(t)=t+1$ and $h(t)=t+2$ then

$$\begin{aligned} f(t) &= t[1-u(t-1)]+(2-t)[u(t-1)-u(t-2)] \\ &= t-2t\,u(t-1)+2u(t-1)-2u(t-2)+t\,u(t-2) \\ &= t-2u(t-1)g(t-1)+2u(t-1)-2u(t-2)+u(t-2)h(t-2) \end{aligned}$$

so

$$F(s) = \frac{1}{s^2}-2e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)+\frac{2e^{-s}}{s}-\frac{2e^{-2s}}{s}+e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) = \frac{\left(1-e^{-s}\right)^2}{s^2}.$$

22. $f(t) = [u_1(t) - u_2(t)]\,t^3 = u_1(t)g(t-1) - u_2(t)h(t-2)$ where

$$\begin{aligned} g(t) &= (t+1)^3 = t^3+3t^2+3t+1, \\ h(t) &= (t+2)^3 = t^3+6t^2+12t+8. \end{aligned}$$

It follows that

$$\begin{aligned} F(s) &= e^{-s}G(s) - e^{-2s}H(s) \\ &= [(s^3+3s^2+6s+6)e^{-s} - (8s^3+12s^2+12s+6)e^{-2s}]/s^4. \end{aligned}$$

23. With $f(t) = 1$ and $p=1$, Formula (12) in the text gives

$$\mathcal{L}\{1\} = \frac{1}{1-e^{-s}}\int_0^1 e^{-st}\cdot 1\,dt = \frac{1}{1-e^{-s}}\left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=1} = \frac{1}{s}.$$

24. With $f(t) = \cos kt$ and $p = 2\pi/k$, Formula (12) and the integral formula

$$\int e^{at}\cos bt\,dt = e^{at}\left[\frac{a\cos bt + b\sin bt}{a^2+b^2}\right]+C$$

give

$$\begin{aligned}\mathcal{L}\{\cos kt\} &= \frac{1}{1-e^{-2\pi s/k}}\int_0^{2\pi/k} e^{-st}\cdot\cos kt\,dt\\ &= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-st}\left(\frac{-s\cos kt + k\sin kt}{s^2+k^2}\right)\right]_{t=0}^{t=2\pi/k}\\ &= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-2\pi s/k}\left(\frac{-s}{s^2+k^2}\right)-e^{-0}(-s)\right] = \frac{s}{s^2+k^2}.\end{aligned}$$

25. With $p = 2a$ and $f(t) = 1$ if $0 \le t \le a$, $f(t) = 0$ if $a < t \le 2a$, Formula (12) gives

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2as}}\int_0^a e^{-st}\cdot 1\,dt = \frac{1}{1-e^{-2as}}\left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=a}\\ &= \frac{1-e^{-as}}{s\left(1-e^{-as}\right)\left(1+e^{-as}\right)} = \frac{1}{s\left(1+e^{-as}\right)}.\end{aligned}$$

26. With $p = a$ and $f(t) = t/a$, Formula (12) and the integral formula $\int ue^u\,du = (u-1)e^u$ (with $u = -st$) give

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{a\left(1-e^{-as}\right)}\int_0^a e^{-st}\cdot t\,dt = \frac{1}{a\left(1-e^{-as}\right)}\int_0^{-as} e^u\cdot\left(-\frac{u}{s}\right)\left(-\frac{du}{s}\right)\\ &= \frac{1}{as^2\left(1-e^{-as}\right)}\int_0^{-as} e^u u\,du = \frac{1}{as^2\left(1-e^{-as}\right)}\left[(u-1)e^u\right]_0^{-as}\\ &= \frac{1}{as^2\left(1-e^{-as}\right)}\left[(-as-1)e^{-as}+1\right] = \frac{1}{as^2}-\frac{e^{-as}}{s\left(1-e^{-as}\right)}.\end{aligned}$$

27. $G(s) = \mathcal{L}\{t/a - f(t)\} = (1/as^2) - F(s)$. Now substitution of the result of Problem 26 in place of $F(s)$ immediately gives the desired transform.

28. This computation is very similar to the one in Problem 26, except that $p = 2a$:

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2as}}\int_0^a e^{-st}\cdot t\,dt = \frac{1}{1-e^{-2as}}\int_0^{-as} e^u\cdot\left(-\frac{u}{s}\right)\left(-\frac{du}{s}\right)\\ &= \frac{1}{s^2\left(1-e^{-2as}\right)}\int_0^{-as} e^u u\,du = \frac{1}{s^2\left(1-e^{-2as}\right)}\left[(u-1)e^u\right]_0^{-as}\end{aligned}$$

$$= \frac{1}{s^2\left(1-e^{-2as}\right)}\left[\left(-as-1\right)e^{-as}+1\right] = \frac{1-e^{-as}(1+as)}{s^2\left(1-e^{-2as}\right)}.$$

29. With $p = 2\pi/k$ and $f(t) = \sin kt$ for $0 \le t \le \pi/k$ while $f(t) = 0$ for $\pi/k \le t \le 2\pi/k$, Formula (12) and the integral formula

$$\int e^{at}\sin bt\,dt = e^{at}\left[\frac{a\sin bt - b\cos bt}{a^2+b^2}\right]+C$$

give

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2\pi s/k}}\int_0^{\pi/k} e^{-st}\cdot \sin kt\,dt \\
&= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-st}\left(\frac{-s\sin kt - k\cos kt}{s^2+k^2}\right)\right]_{t=0}^{t=\pi/k} \\
&= \frac{1}{1-e^{-2\pi s/k}}\left[\frac{e^{-\pi s/k}\left(k\right)-(-k)}{s^2+k^2}\right] \\
&= \frac{k\left(1+e^{-\pi s/k}\right)}{\left(1-e^{-\pi s/k}\right)\left(1+e^{-\pi s/k}\right)\left(s^2+k^2\right)} = \frac{k}{\left(s^2+k^2\right)\left(1-e^{-\pi s/k}\right)}.
\end{aligned}$$

30. $h(t) = f(t)+g(t) = f(t)+u(t-\pi/k)f(t-\pi/k)$, so Problem 29 gives

$$\begin{aligned}
H(s) &= F(s)+e^{-\pi s/k}F(s) = \left(1+e^{-\pi s/k}\right)F(s) \\
&= \left(1+e^{-\pi s/k}\right)\cdot\frac{k}{\left(s^2+k^2\right)\left(1-e^{-\pi s/k}\right)} = \frac{k}{s^2+k^2}\cdot\frac{1+e^{-\pi s/k}}{1-e^{-\pi s/k}}\cdot\frac{e^{\pi s/2k}}{e^{\pi s/2k}} \\
&= \frac{k}{s^2+k^2}\cdot\frac{e^{\pi s/2k}+e^{-\pi s/2k}}{e^{\pi s/2k}-e^{-\pi s/2k}} = \frac{k}{s^2+k^2}\frac{\cosh(\pi s/2k)}{\sinh(\pi s/2k)} = \frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}.
\end{aligned}$$

In Problems 31-42, we first write and transform the appropriate differential equation. Then we solve for the transform of the solution, and finally inverse transform to find the desired solution.

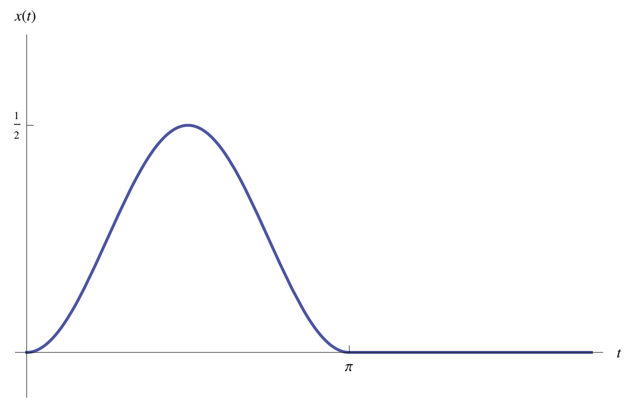
31. $x'' + 4x = 1 - u(t-\pi)$

$$s^2X(s) + 4X(s) = \frac{1-e^{-\pi s}}{s}$$

$$X(s) = \frac{1-e^{-\pi s}}{s\left(s^2+4\right)} = \frac{1}{4}\left(1-e^{-\pi s}\right)\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$x(t) = (1/4)[1 - u(t-\pi)][1 - \cos 2(t-\pi)] = (1/2)[1 - u(t-\pi)]\sin^2 t$$

The graph of the position function $x(t)$ is shown at the top of the next page.



32. $x'' + 5x' + 4x = 1 - u(t-2)$

$$s^2X(s) + 5s\,X(s) + 4X(s) = \frac{1-e^{-2s}}{s}$$

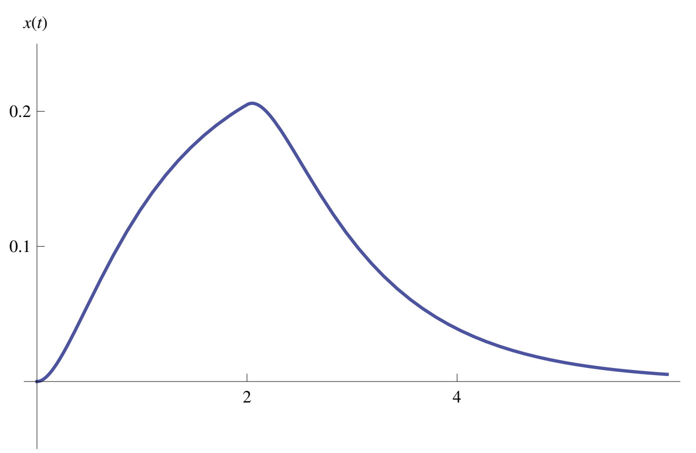
$$X(s) = \frac{1-e^{-2s}}{s\left(s^2+5s+4\right)} = (1 - e^{-2s})G(s)$$

where

$$G(s) = \frac{1}{12}\left(\frac{3}{s} - \frac{4}{s+1} + \frac{1}{s+4}\right), \text{ so } g(t) = \frac{1}{12}\left(3 - 4e^{-t} + e^{-4t}\right).$$

It follows that

$$x(t) = g(t) - u(t-2)g(t-2) = \begin{cases} g(t) & \text{if } t<2, \\ g(t)-g(t-2) & \text{if } t \geq 2. \end{cases}$$

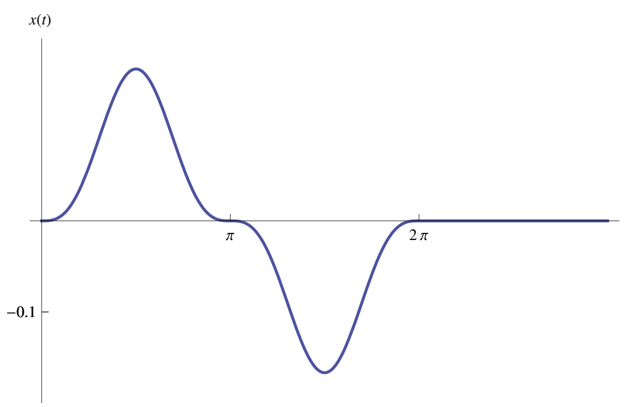


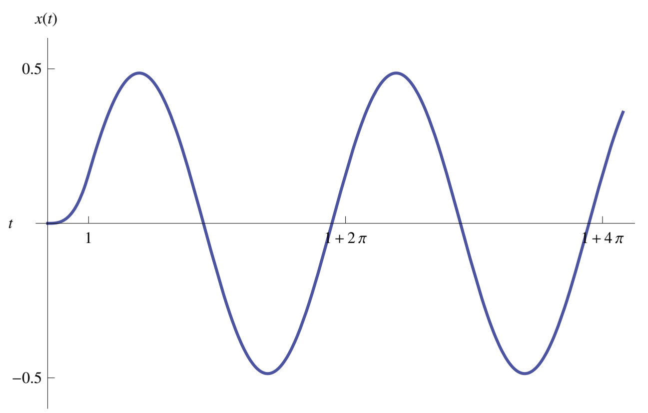
33. $x'' + 9x = [1 - u(t-2\pi)]\sin t$

$$X(s) = \frac{1-e^{-2\pi s}}{(s^2+1)(s^2+9)} = \frac{1}{8}\left(1-e^{-2\pi s}\right)\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right)$$

$$x(t) = \frac{1}{8}[1-u(t-2\pi)]\left(\sin t-\frac{1}{3}\sin 3t\right)$$

The left-hand figure below show the graph of this position function.





34. $x''+x = [1 - u(t-1)]\,t = 1-u(t-1)f(t-1)$, where $f(t) = t+1$

$$s^2X(s)+X(s) = \frac{1}{s^2}-e^{-s}G(s) = \frac{1}{s^2}-e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

It follows that

$$\begin{aligned} X(s) &= \frac{1}{s^2(s^2+1)}-\frac{e^{-s}(s+1)}{s^2(s^2+1)} \\ &= (1-e^{-s})\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)-e^{-s}\left(\frac{1}{s}-\frac{s}{s^2+1}\right) = (1-e^{-s})G(s)-e^{-s}H(s) \end{aligned}$$

where $g(t) = t - \sin t$, $h(t) = 1 - \cos t$. Hence

$$x(t) = g(t) - u(t-1)g(t-1) - u(t-1)h(t-1)$$

and so

$$\begin{aligned} x(t) &= t - \sin t \ \text{ if } t<1, \\ x(t) &= -\sin t + \sin(t-1) + \cos(t-1) \ \text{ if } t>1. \end{aligned}$$

The right-hand figure above shows the graph of this position function.

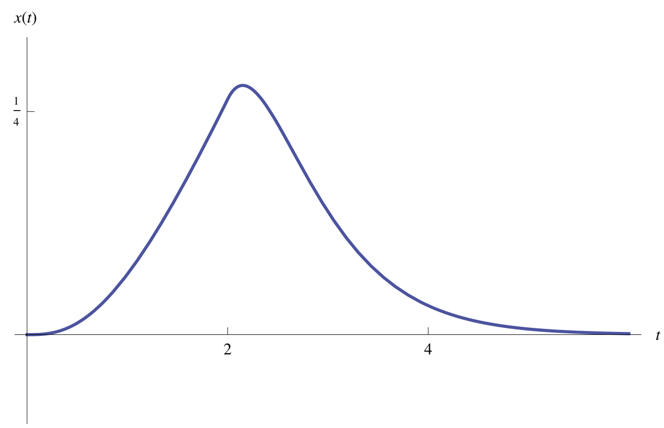
35. $x'' + 4x' + 4x = [1 - u(t-2)]t = t - u(t-2)g(t-2)$ where $g(t)=t+2$

$$(s+2)^2X(s) = \frac{1}{s^2}-e^{-2s}\left(\frac{2}{s}+\frac{1}{s^2}\right)$$

$$X(s) = \frac{1}{s^2(s+2)^2} - e^{-2s}\frac{2s+1}{s^2(s+2)^2}$$

$$= \frac{1}{4}\left(-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+2}+\frac{1}{(s+2)^2}\right) - \frac{1}{4}e^{-2s}\left(\frac{1}{s}+\frac{1}{s^2}-\frac{1}{s+2}-\frac{3}{(s+2)^2}\right)$$

$$x(t) = (1/4)\{-1 + t + (1+t)e^{-2t} + u(t-2)[1 - t + (3t-5)e^{-2(t-2)}]\}$$



36. $$100\,I(s) + 1000\frac{I(s)}{s} = 100\left(\frac{1}{s} - \frac{e^{-s}}{s}\right)$$

$$I(s) = \frac{1-e^{-s}}{s+10} = \left(1-e^{-s}\right)\mathcal{L}\{e^{-10t}\}$$

$$i(t) = e^{-10t} - u(t-1)e^{-10(t-1)}$$

37. $$i'(t) + 10^4\int i(t)\,dt = 100[1 - u(t-2\pi)]$$

$$s\,I(s) + 10^4\frac{I(s)}{s} = 100\frac{1-e^{-2\pi s}}{s}$$

$$I(s) = \frac{100\left(1-e^{-2\pi s}\right)}{s^2+10^4} = \left(1-e^{-2\pi s}\right)\mathcal{L}\{\sin 100t\}$$

$$i(t) = \sin 100t - u(t-2\pi)\sin 100(t-2\pi) = [1 - u(t-2\pi)]\sin 100t$$

38. $$i'(t) + 10000\int i(t)\,dt = [1 - u(t-\pi)](100\sin 10t)$$

$$s\,I(s) + 10000\,I(s)/s = 1000(1 - e^{-\pi s})/(s^2+100)$$

$$I(s) = \left(1-e^{-\pi s}\right)\cdot\frac{1000s}{\left(s^2+100\right)\left(s^2+10000\right)} = \left(1-e^{-\pi s}\right)\cdot\frac{10}{99}\left(\frac{s}{s^2+10^2}-\frac{s}{s^2+100^2}\right)$$

$$= \frac{10}{99}\left(1-e^{-\pi s}\right)\mathcal{L}\{\cos 10t - \cos 100t\}$$

$$i(t) = \frac{10}{99}\left(\cos 10t - \cos 100t\right) - \frac{10}{99}u\left(t-\pi\right)\left[\cos 10(t-\pi) - \cos 100(t-\pi)\right]$$

$$= \frac{10}{99}\left[1-u\left(t-\pi\right)\right]\left(\cos 10t - \cos 100t\right)$$

$$i(t) = \begin{cases} \frac{10}{99}\left(\cos 10t - \cos 100t\right) & \text{if } t \le \pi, \\ 0 & \text{if } t > \pi \end{cases}$$

39. $i'(t) + 150\, i(t) + 5000\int i(t)\, dt = 100t[1 - u(t-1)]$

$$sI(s) + 150I(s) + 5000\frac{I(s)}{s} = \frac{100}{s^2} - 100e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

$$I(s) = \frac{100}{s(s+50)(s+100)} - e^{-s}\cdot\frac{100(s+1)}{s(s+50)(s+100)}$$

$$= \frac{1}{50}\left(\frac{1}{s}-\frac{2}{s+50}+\frac{1}{s+100}\right) - \frac{1}{50}e^{-s}\left(\frac{1}{s}+\frac{98}{s+50}-\frac{99}{s+100}\right)$$

$$i(t) = (1/50)[1 - 2e^{-50t} + e^{-100t}] - (1/50)u(t-1)[1 + 98e^{-50(t-1)} - 99e^{-100(t-1)}]$$

40. $i'(t) + 100\, i(t) + 2500\int i(t)\, dt = 50t[1 - u(t-1)]$

$$sI(s) + 100I(s) + 2500\frac{I(s)}{s} = \frac{50}{s^2} - 50e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

$$I(s) = \frac{50}{s(s+50)^2} - e^{-s}\cdot\frac{50(s+1)}{s(s+50)^2}$$

$$= \frac{1}{50}\left(\frac{1}{s}-\frac{1}{s+50}-\frac{50}{(s+50)^2}\right) - \frac{1}{50}e^{-s}\left(\frac{1}{s}-\frac{50}{s+50}+\frac{2450}{(s+50)^2}\right)$$

$$i(t) = \frac{1}{50}\left(1-e^{-50t}-50t\,e^{-50t}\right) - \frac{1}{50}u(t-1)\left(1-e^{-50(t-1)}+2450t\,e^{-50(t-1)}\right)$$

41. $x'' + 4x = f(t), \qquad x(0) = x'(0) = 0$

$$\left(s^2+4\right)X(s) = \frac{4\left(1-e^{-\pi s}\right)}{s\left(1+e^{-\pi s}\right)} \qquad \text{(by Example 6 of Section 7.5)}$$

$$\left(s^2+4\right)X(s) = \frac{4}{s}+\frac{8}{s}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s} \qquad \text{(as in Eq. (16) of Section 7.5)}$$

Now let

$$g(t) = \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)}\right\} = 1-\cos 2t = 2\sin^2 t.$$

Then it follows that

$$x(t) = g(t)+2\sum_{n=1}^{\infty}(-1)^n u_{n\pi}(t)g(t-n\pi) = 2\sin^2 t+4\sum_{n=1}^{\infty}(-1)^n u_{n\pi}(t)\sin^2 t.$$

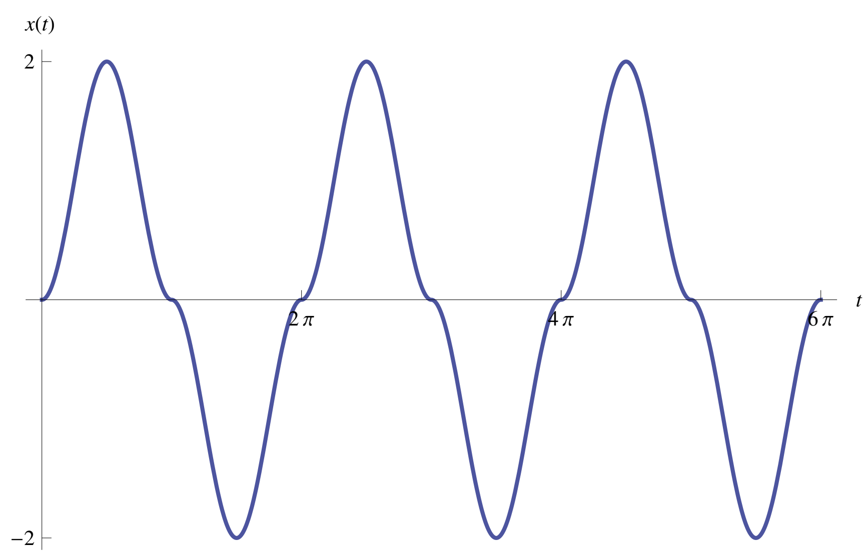
Hence

$$x(t) = \begin{cases} 2\sin^2 t & \text{if } 2n\pi \le t < (2n+1)\pi, \\ -2\sin^2 t & \text{if } (2n-1)\pi \le t < 2n\pi. \end{cases}$$

Consequently the complete solution

$$x(t) = 2\left|\sin t\right|\sin t$$

is periodic, so the transient solution is zero. The graph of $x(t)$:



42. $x'' + 2x' + 10x = f(t), \quad x(0) = x'(0) = 0$

As in the solution of Example 8 we find first that

$$\left(s^2 + 2s + 10\right) X(s) = \frac{10}{s} + \frac{20}{s} \sum_{n=1}^{\infty} (-1)^n e^{-n\pi s},$$

so

$$X(s) = \frac{10}{s(s^2 + 2s + 10)} + 2\sum_{n=1}^{\infty} \frac{10(-1)^n e^{-n\pi s}}{s(s^2 + 2s + 10)}.$$

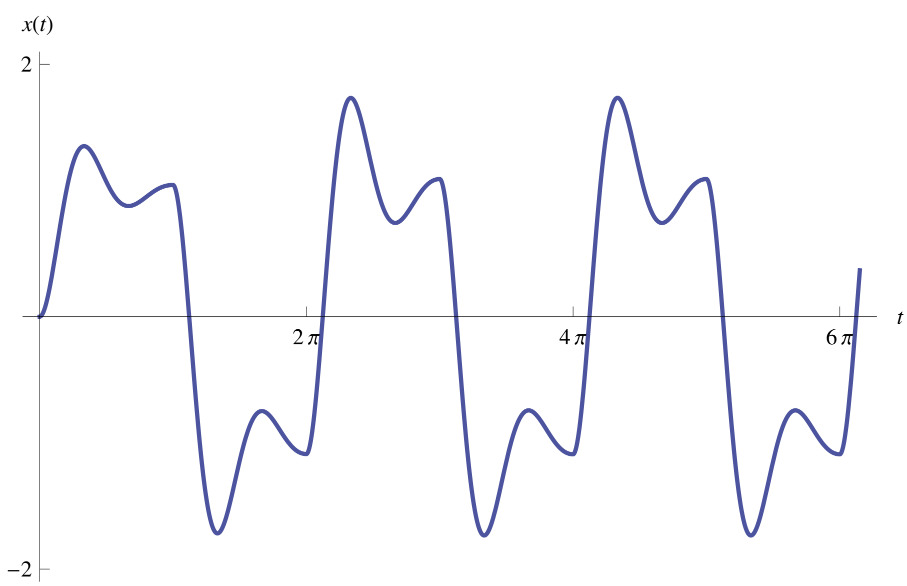
If

$$g(t) = \mathcal{L}^{-1}\left\{\frac{10}{s\left[(s+1)^2 + 9\right]}\right\} = 1 - \frac{1}{3} e^{-t}\left(3\cos 3t + \sin 3t\right),$$

then it follows that

$$x(t) = g(t) + 2\sum_{n=1}^{\infty} (-1)^n u_{n\pi}(t) g(t - n\pi).$$

The graph of $x(t)$:



SECTION 7.6

IMPULSES AND DELTA FUNCTIONS

Among the several ways of introducing delta functions, we consider the physical approach of the first two pages of this section to be the most tangible one for elementary students. Whatever the approach, however, the practical consequences are the same — as described in the discussion associated with equations (11)–(19) in the text. That is, in order to solve a differential equation of the form

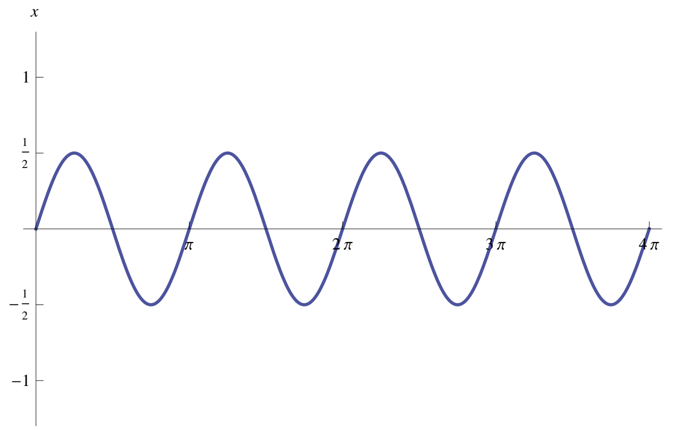
$$a\,x''(t) + b\,x'(t) + c\,x(t) \;=\; f(t)$$

where $f(t)$ involves delta functions, we transform the equation using the operational principle $\mathcal{L}\{\delta_a(t)\} = e^{-as}$, then solve for $X(s)$, and finally invert as usual to find the formal solution $x(t)$. Then we show the graph of $x(t)$.

1. $s^2X(s) + 4X(s) = 1$

$$X(s) = \frac{1}{s^2+4}$$

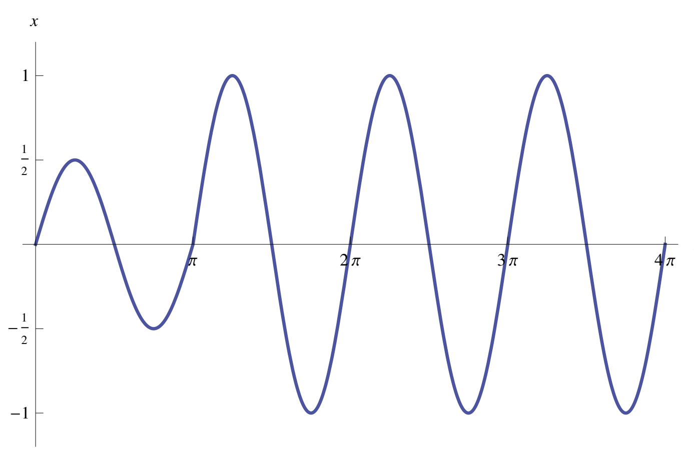
$$x(t) = \frac{1}{2}\sin 2t$$



2. $s^2X(s) + 4X(s) = 1 + e^{-\pi s}$

$$X(s) = \frac{1+e^{-\pi s}}{s^2+4}$$

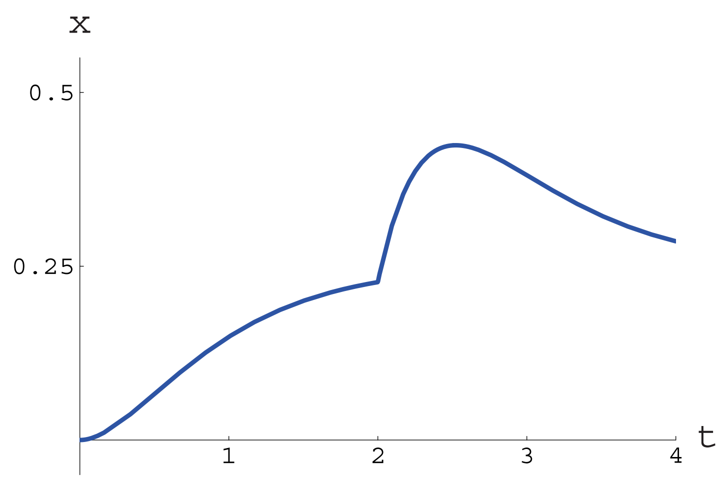
$$x(t) = \frac{1}{2}[1+u(t-\pi)]\sin 2t = \begin{cases} \dfrac{1}{2}\sin 2t & \text{if } t \le \pi, \\ \sin 2t & \text{if } t > \pi. \end{cases}$$



3. $s^2X(s) + 4sX(s) + 4X(s) = \frac{1}{s} + e^{-2s}$

$$X(s) = \frac{1}{s(s+2)^2} + \frac{e^{-2s}}{(s+2)^2} = \frac{1}{4}\left(\frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}\right) + \frac{e^{-2s}}{(s+2)^2}$$

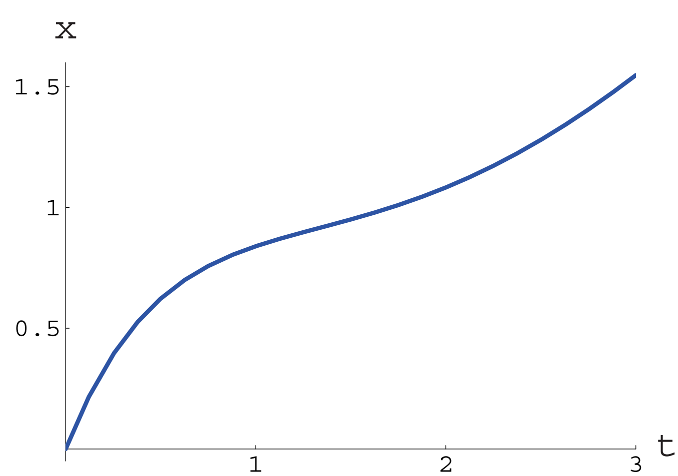
$$x(t) = \frac{1}{4}\left[1 - e^{-2t} - 2te^{-2t}\right] + u(t-2)(t-2)e^{-2(t-2)}$$



4. $[s^2X(s) - 1] + 2sX(s) + X(s) = 1 + \frac{1}{s^2}$

$$X(s) = \frac{2s^2+1}{s^2(s+1)^2} = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2}$$

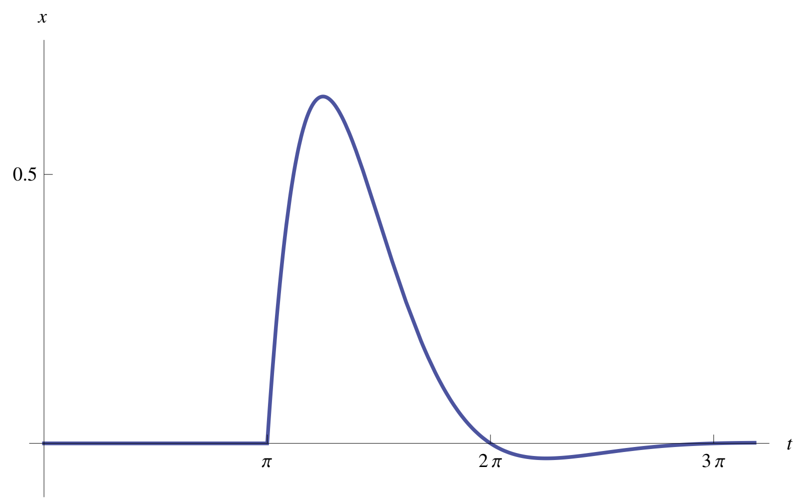
$$x(t) = -2 + t + 2e^{-t} + 3te^{-t}$$



5. $(s^2 + 2s + 2)X(s) = 2e^{-\pi s}$

$$X(s) = \frac{2e^{-\pi s}}{(s+1)^2 + 1}$$

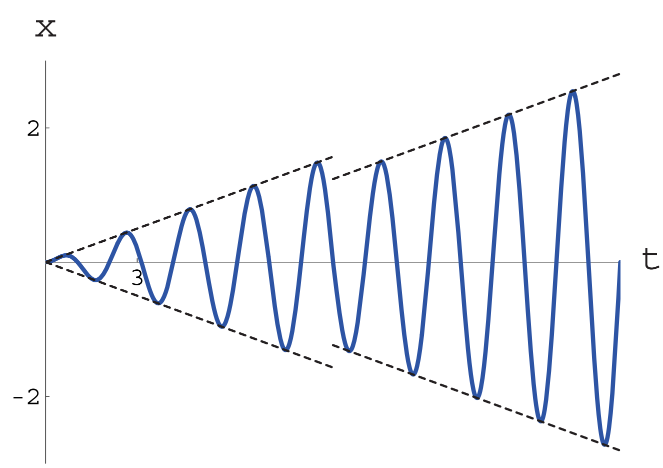
$$x(t) = 2u(t-\pi)e^{-(t-\pi)}\sin(t - \pi) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \pi, \\ -2e^{-(t-\pi)}\sin t & \text{if} \quad t \ge \pi. \end{cases}$$



6. $s^2X(\mathrm{s}) + 9X(s) = e^{-3\pi s} + \dfrac{s}{s^2+9}$

$$X(s) = \frac{s}{\left(s^2+9\right)^2} + \frac{e^{-3\pi s}}{s^2+9}$$

$$x(t) = \frac{1}{6}t\sin 3t + \frac{1}{3}u(t-3\pi)\sin 3(t-3\pi) = \frac{1}{6}t\sin 3t - \frac{1}{3}u(t-3\pi)\sin 3t$$

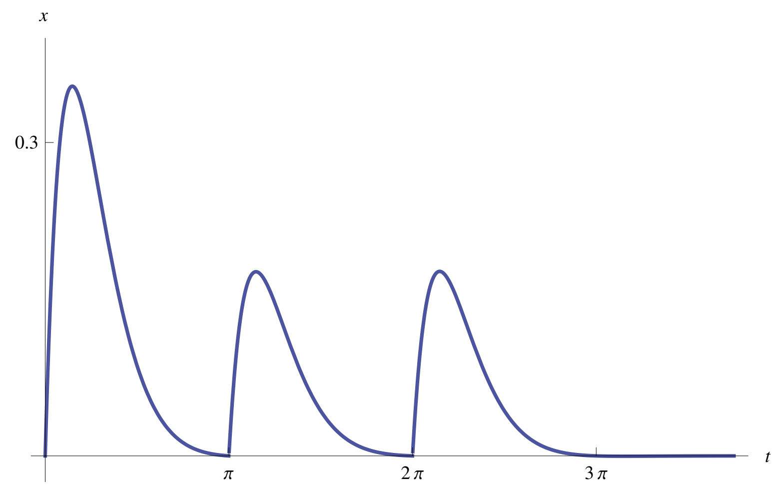


7. $[s^2X(s) - 2] + 4sX(s) + 5X(s) = e^{-\pi s} + e^{-2\pi s}$

$$X(s) = \frac{2 + e^{-\pi s} + e^{-2\pi s}}{(s+2)^2 + 1}$$

$$x(t) = 2e^{-2t}\sin t + u_\pi(t)e^{-2(t-\pi)}\sin(t - \pi) + u_{2\pi}(t)e^{-2(t-2\pi)}\sin(t - 2\pi)$$

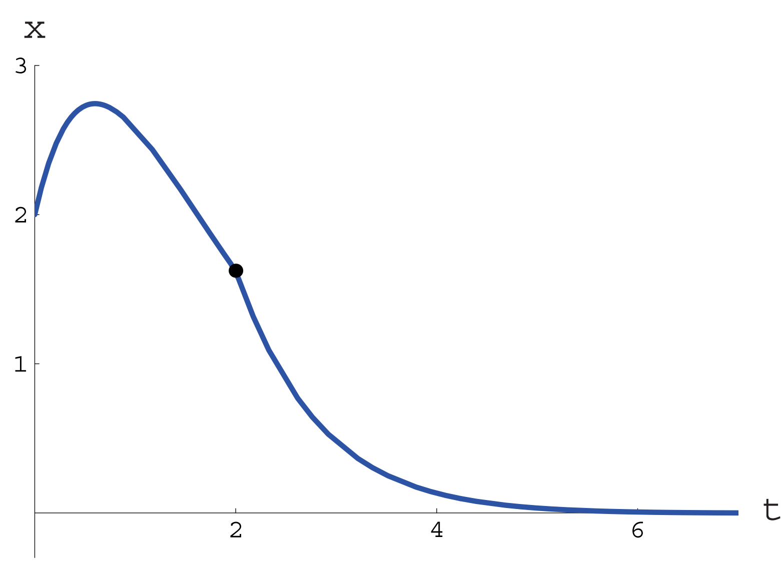
$$= [2 - e^{2\pi}u(t-\pi) + e^{4\pi}u(t-2\pi)]\ e^{-2t}\sin t$$



8. $[s^2X(s) - 2s - 2] + 2[sX(s) - 2] + X(s) = 1 - e^{-2s}$

$$X(s) = \frac{2s + 7 - e^{-2s}}{(s+1)^2} = \frac{2}{s+1} + \frac{5}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}$$

$$x(t) = (2 + 5t)e^{-t} - u(t-2)(t - 2)e^{-(t-2)}$$



9. $s^2X(s) + 4X(s) = F(s)$

$$X(s) = \frac{1}{s^2+4}\cdot F(s)$$

$$x(t) = \tfrac{1}{2}\int_0^t (\sin 2u)\, f(t-u)\, du$$

10. $s^2X(s) + 6s\,X(s) + 9X(s) = F(s)$

$$X(s) = \frac{1}{(s+3)^2}\cdot F(s)$$

$$x(t) = \int_0^t ue^{-3u} f(t-u)\, du$$

11. $(s^2 + 6s + 8)X(s) = F(s)$

$$X(s) = \frac{1}{(s+3)^2-1}\cdot F(s)$$

$$x(t) = \int_0^t e^{-3u}(\sinh u) f(t-u)\, du$$

12. $s^2X(s) + 4sX(s) + 8X(s) = F(s)$

$$X(s) = \frac{1}{(s+2)^2+4}\cdot F(s)$$

$$x(t) = \tfrac{1}{2}\int_0^t e^{-2u}(\sin 2u) f(t-u)\, du$$

13. **(a)** $mx_\varepsilon''(t) = (p/\varepsilon)[u_0(t) - u_\varepsilon(t)]$

$$ms^2X_\varepsilon(s) = (p/\varepsilon)[1/s - e^{-\varepsilon s}/s]$$

$$mX_\varepsilon(s) = (p/\varepsilon)[(1 - e^{-\varepsilon s})/s^3]$$

$$mx_\varepsilon(t) = (p/2\varepsilon)[t^2 - u_\varepsilon(t)(t - \varepsilon)^2]$$

(b) If $t \geq \varepsilon$ then

$$mx_\varepsilon(t) = (p/2\varepsilon)[t^2 - (t^2 - 2\varepsilon t + \varepsilon^2)] = (p/2\varepsilon)(2\varepsilon t - \varepsilon^2).$$

Hence $mx_\varepsilon(t) \to pt$ as $\varepsilon \to 0$.

(c) $mv = (mx)' = (pt)' = p.$

14. $sX(s) = e^{-as}; \quad X(s) = e^{-as}/s; \quad x(t) = u(t - a)$

15. Each of the two given initial value problems transforms to

$$(ms^2 + k)X(s) = mv_0 = p_0.$$

16. Each of the two given initial value problems transforms to

$$(as^2 + bs + c)X(s) = F(s) + av_0$$

17. **(b)** $i' + 100i = \delta_1(t) - \delta_2(t), \quad i(0) = 0$

$$I(s) = \frac{e^{-s} - e^{-2s}}{s+100} I(s)$$

$$i(t) = u_1(t)e^{-100(t-1)} - u_2(t)e^{-100(t-2)}$$

18. **(b)** $i''(t) + 100\, i(t) = 10\, \delta(t) - 10\, \delta(t - \pi)$

$$(s^2 + 100)\, I(s) = 10 - 10\, e^{-\pi s}$$

$$I(s) = \frac{10}{s^2+100} - \frac{10e^{-\pi s}}{s^2+100}$$

$$i(t) = \sin 10t - u_\pi(t)\sin 10(t - \pi)$$

$$= [1 - u(t-\pi)]\sin 10t = \begin{cases} \sin 10t & \text{if } t \le \pi, \\ 0 & \text{if } t \ge \pi \end{cases}$$

19. $\left(s^2+100\right)I(s) = 10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/10}$

$$I(s) = \frac{10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/10}}{s^2+100} = \sum_{n=0}^{\infty}\left((-1)^n e^{-n\pi s/10}\cdot\frac{10}{s^2+100}\right)$$

$$i(t) = \sum_{n=0}^{\infty}(-1)^n u_{n\pi/10}(t)\sin 10(t - n\pi/10) = \sum_{n=0}^{\infty} u(t - n\pi/10)\sin 10t$$

because $\sin(10t - n\pi) = (-1)^n \sin 10t$. Hence

$$i(t) = (n+1)\sin 10t$$

if $n\pi/10 < t < (n+1)\pi/10$.

20. $\left(s^2+100\right)I(s) = 10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/5}$

$$I(s) = \frac{10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/5}}{s^2+100} = \sum_{n=0}^{\infty}\left((-1)^n e^{-n\pi s/5}\cdot\frac{10}{s^2+100}\right)$$

$$i(t) = \sum_{n=0}^{\infty}(-1)^n u_{n\pi/5}(t)\sin 10(t-n\pi/5) = \sum_{n=0}^{\infty}(-1)^n u(t-n\pi/5)\sin 10t$$

Hence

$$i(t) = \sin 10t + (-1)^1\sin 10t + \cdots + (-1)^n\sin 10t$$

if $n\pi/5 < t < (n+1)\pi/5,\ n\geq 0$. Thus $i(t)=\sin 10t$ in this interval if n is even, but is zero in this interval if n is odd.

21. $\left(s^2+60s+1000\right)I(s) = 10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/10}$

$$I(s) = \frac{10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/10}}{s^2+60s+1000} = \sum_{n=0}^{\infty}\left((-1)^n e^{-n\pi s/10}\cdot\frac{10}{(s+30)^2+100}\right)$$

$i(t) = \Sigma\,(-1)^n u_{n\pi/10}(t)\, g(t - n\pi/10)$

where $g(t) = e^{-30t}\sin 10t$, and so

$$\begin{aligned} g(t - n\pi/10) &= \exp[-30(t - n\pi/10)]\sin 10(t - n\pi/10) \\ &= e^{3n\pi}e^{-30t}\cdot(-1)^n\sin 10t \end{aligned}$$

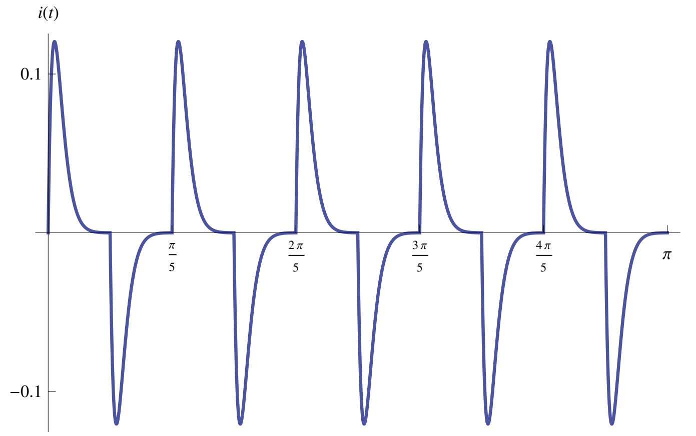
Therefore

$$i(t) = \sum_{n=0}^{\infty}u\left(t-\frac{n\pi}{10}\right)e^{3n\pi}e^{-30t}\sin 10t\,.$$

If $n\pi/10 < t < (n+1)\pi/10$ then it follows that

$$i(t) = (1 + e^{3\pi} + \cdots + e^{3n\pi})e^{-30t}\sin 10t = \frac{e^{3(n+1)\pi}-1}{e^{3\pi}-1}e^{-30t}\sin 10t\,.$$

The graph of $i(t)$ is shown at the top of the next page.



22. $(s^2+1)X(s) = \sum_{n=0}^{\infty} e^{-2n\pi s}$

$$X(s) = \sum_{n=0}^{\infty} \frac{e^{-2n\pi s}}{s^2+1}$$

$$x(t) = \sum_{n=0}^{\infty} u_{2n\pi}(t)\sin\left(t-2n\pi\right) = \sum_{n=0}^{\infty} u(t-2n\pi)\sin t$$

Hence $x(t) = (n+1)\sin t$ if $2n\pi < t < 2(n+1)\pi$. The graph of $x(t)$: