CHAPTER 8

POWER SERIES METHODS

SECTION 8.1

INTRODUCTION AND REVIEW OF POWER SERIES

The power series method consists of substituting a series $y = \Sigma c_n x^n$ into a given differential equation in order to determine what the coefficients $\{c_n\}$ must be in order that the power series will satisfy the equation. It might be pointed out that, if we find a recurrence relation in the form $c_{n+1} = \phi(n)c_n$, then we can determine the radius of convergence ρ of the series solution directly from the recurrence relation

$$\rho = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{\phi(n)}\right|.$$

In Problems 1–10 we give first a recurrence relation that can be used to find the radius of convergence and to calculate the succeeding coefficients $c_1, c_2, c_3, \cdots$ in terms of the arbitrary constant c_0. Then we give the series itself.

1. $c_{n+1} = \dfrac{c_n}{n+1}$; it follows that $c_n = \dfrac{c_0}{n!}$ and $\rho = \lim_{n\to\infty}(n+1) = \infty$.

$$y(x) = c_0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) = c_0\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = c_0 e^x$$

2. $c_{n+1} = \dfrac{4c_n}{n+1}$; it follows that $c_n = \dfrac{4^n c_0}{n!}$ and $\rho = \lim_{n\to\infty}\dfrac{n+1}{4} = \infty$.

$$y(x) = c_0\left(1 + 4x + 8x^2 + \frac{32x^3}{3} + \frac{32x^4}{4} + \cdots\right)$$

$$= c_0\left(1 + \frac{4x}{1!} + \frac{4^2x^2}{2!} + \frac{4^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots\right) = c_0 e^{4x}$$

3. $c_{n+1} = -\dfrac{3c_n}{2(n+1)}$; it follows that $c_n = \dfrac{(-1)^n 3^n c_0}{2^n n!}$ and $\rho = \lim_{n\to\infty}\dfrac{2(n+1)}{3} = \infty$.

$$y(x) = c_0\left(1 - \frac{3x}{2} + \frac{9x^2}{8} - \frac{9x^3}{16} + \frac{27x^4}{128} - \cdots\right)$$

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$$= c_0\left(1-\frac{3x}{1!2}+\frac{3^2x^2}{2!2^2}-\frac{3^3x^3}{3!2^3}+\frac{3^4x^4}{4!2^4}-\cdots\right) = c_0e^{-3x/2}$$

4. When we substitute $y = \Sigma c_n x^n$ into the equation $y' + 2xy = 0$, we find that

$$c_1+\sum_{n=0}^{\infty}\left[(n+2)c_{n+2}+2c_n\right]x^{n+1} = 0.$$

Hence $c_1 = 0$ — which we see by equating constant terms on the two sides of this equation — and $c_{n+2} = -\dfrac{2c_n}{n+2}$. It follows that

$$c_1 = c_3 = c_5 = \cdots = c_{\text{odd}} = 0 \quad \text{and} \quad c_{2k} = \frac{(-1)^k c_0}{k!}.$$

Hence

$$y(x) = c_0\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{3}+\cdots\right) = c_0\left(1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots\right) = c_0e^{-x^2}$$

and $\rho = \infty$.

5. When we substitute $y = \Sigma c_n x^n$ into the equation $y' = x^2 y$, we find that

$$c_1+2c_2x+\sum_{n=0}^{\infty}\left[(n+3)c_{n+3}-c_n\right]x^{n+2} = 0.$$

Hence $c_1 = c_2 = 0$ — which we see by equating constant terms and x-terms on the two sides of this equation — and $c_{n+3} = \dfrac{c_n}{n+3}$. It follows that

$$c_{3k+1} = c_{3k+2} = 0 \quad \text{and} \quad c_{3k} = \frac{c_0}{3\cdot 6\cdot\cdots\cdot(3k)} = \frac{c_0}{k!3^k}.$$

Hence

$$y(x) = c_0\left(1+\frac{x^3}{3}+\frac{x^6}{18}+\frac{x^9}{162}+\cdots\right) = c_0\left(1+\frac{x^3}{1!3}+\frac{x^6}{2!3^2}+\frac{x^9}{3!3^3}+\cdots\right) = c_0e^{(x^3/3)}$$

and $\rho = \infty$.

6. $c_{n+1} = \dfrac{c_n}{2}$; it follows that $c_n = \dfrac{c_0}{2^n}$ and $\rho = \lim\limits_{n\to\infty} 2 = 2.$

$$y(x) = c_0\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{16}+\cdots\right)$$

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$$= c_0\left[1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+\cdots\right] = \frac{c_0}{1-\frac{x}{2}} = \frac{2c_0}{2-x}$$

7. $c_{n+1} = 2c_n$; it follows that $c_n = 2^n c_0$ and $\rho = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}$.

$$y(x) = c_0\left(1+2x+4x^2+8x^3+16x^4+\cdots\right)$$
$$= c_0\left[1+(2x)+(2x)^2+(2x)^3+(2x)^4+\cdots\right] = \frac{c_0}{1-2x}$$

8. $c_{n+1} = -\frac{(2n-1)c_n}{2n+2}$; it follows that $\rho = \lim_{n\to\infty}\frac{2n+2}{2n-1} = 1$.

$$y(x) = c_0\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots\right)$$

Separation of variables gives $y(x) = c_0\sqrt{1+x}$.

9. $c_{n+1} = \frac{(n+2)c_n}{n+1}$; it follows that $c_n = (n+1)c_0$ and $\rho = \lim_{n\to\infty}\frac{n+1}{n+2} = 1$.

$$y(x) = c_0\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)$$

Separation of variables gives $y(x) = \frac{c_0}{(1-x)^2}$.

10. $c_{n+1} = \frac{(2n-3)c_n}{2n+2}$; it follows that $\rho = \lim_{n\to\infty}\frac{2n+2}{2n-3} = 1$.

$$y(x) = c_0\left(1-\frac{3x}{2}+\frac{3x^2}{8}+\frac{x^3}{16}+\frac{3x^4}{128}+\cdots\right)$$

Separation of variables gives $y(x) = c_0(1-x)^{3/2}$.

In Problems 11–14 the differential equations are second-order, and we find that the two initial coefficients c_0 and c_1 are both arbitrary. In each case we find the even-degree coefficients in terms of c_0 and the odd-degree coefficients in terms of c_1. The solution series in these problems are all recognizable power series that have infinite radii of convergence.

11. $c_{n+1} = \frac{c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \frac{c_0}{(2k)!}$ and $c_{2k+1} = \frac{c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)+c_1\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots\right) = c_0\cosh x + c_1\sinh x$$

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12. $c_{n+1} = \dfrac{4c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \dfrac{2^{2k}c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{2^{2k}c_1}{(2k+1)!}$.

$$\begin{aligned} y(x) &= c_0\left(1+2x^2+\frac{2x^4}{3}+\frac{4x^6}{45}+\cdots\right)+c_1\left(x+\frac{2x^3}{3}+\frac{2x^5}{15}+\frac{4x^7}{315}+\cdots\right) \\ &= c_0\left(1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\frac{(2x)^6}{6!}+\cdots\right)+\frac{c_1}{2}\left((2x)+\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\frac{(2x)^7}{7!}+\cdots\right) \\ &= c_0\cosh 2x+\frac{c_1}{2}\sinh 2x \end{aligned}$$

13. $c_{n+2} = -\dfrac{9c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \dfrac{(-1)^k 3^{2k}c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{(-1)^k 3^{2k}c_1}{(2k+1)!}$.

$$\begin{aligned} y(x) &= c_0\left(1-\frac{9x^2}{2}+\frac{27x^4}{8}-\frac{81x^6}{80}+\cdots\right)+c_1\left(x-\frac{3x^3}{2}+\frac{27x^5}{40}-\frac{81x^7}{560}+\cdots\right) \\ &= c_0\left(1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\frac{(3x)^6}{6!}+\cdots\right)+\frac{c_1}{3}\left((3x)-\frac{(3x)^3}{3!}+\frac{(3x)^5}{5!}-\frac{(3x)^7}{7!}+\cdots\right) \\ &= c_0\cos 3x+\frac{c_1}{3}\sin 3x \end{aligned}$$

14. When we substitute $y = \Sigma c_n x^n$ into $y''+y-x = 0$ and split off the terms of degrees 0 and 1, we get

$$(2c_2+c_0)+(6c_3+c_1-1)\,x+\sum_{n=2}^{\infty}[(n+1)(n+2)c_{n+2}+c_n]x^n = 0.$$

Hence $c_2 = -\dfrac{c_0}{2}$, $c_3 = -\dfrac{c_1-1}{6}$, and $c_{n+2} = -\dfrac{c_n}{(n+1)(n+2)}$ for $n \geq 2$. It follows that

$$\begin{aligned} y(x) &= c_0+c_0\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+c_1x+(c_1-1)\left(-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\ &= x+c_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+(c_1-1)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\ &= x+c_0\cos x+(c_1-1)\sin x. \end{aligned}$$

15. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $xy'+y=0$ and find that $(n+1)c_n = 0$ for all $n \geq 0$. This implies that $c_n = 0$ for all $n \geq 0$, which means that the only power series solution of our differential equation is the trivial solution $y(x) \equiv 0$. Therefore the equation has no *non-trivial* power series solution.

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16. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $2xy' = y$ and find that $2nc_n = c_n$ for all $n \geq 0$. This implies that $0c_0 = c_0,\ 2c_1 = c_1,\ 4c_2 = c_2, \cdots,$ and hence that $c_n = 0$ for all $n \geq 0$, which means that the only power series solution of our differential equation is the trivial solution $y(x) \equiv 0$. Therefore the equation has no *non-trivial* power series solution.

17. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $x^2 y' + y = 0$. We find that $c_0 = c_1 = 0$ and that $c_{n+1} = -nc_n$ for $n \geq 1$, so it follows that $c_n = 0$ for all $n \geq 0$. Just as in Problems 15 and 16, this means that the equation has no *non-trivial* power series solution.

18. When we substitute and assumed power series solution $y = \Sigma c_n x^n$ into $x^3 y' = 2y$, we find that $c_0 = c_1 = c_2 = 0$ and that $c_{n+2} = nc_n/2$ for $n \geq 1$. Hence $c_n = 0$ for all $n \geq 0$, just as in Problems 15–17.

In Problems 19–22 we first give the recurrence relation that results upon substitution of an assumed power series solution $y = \Sigma c_n x^n$ into the given second-order differential equation. Then we give the resulting general solution, and finally apply the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ to determine the desired particular solution.

19. $c_{n+2} = -\dfrac{2^2 c_n}{(n+1)(n+2)}$ for $n \geq 0$, so $c_{2k} = \dfrac{(-1)^k 2^{2k} c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{(-1)^k 2^{2k} c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \cdots\right) + c_1\left(x - \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} - \frac{2^6x^7}{7!} + \cdots\right)$$

$c_0 = y(0) = 0$ and $c_1 = y'(0) = 3$, so

$$\begin{aligned} y(x) &= 3\left(x - \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} - \frac{2^6x^7}{7!} + \cdots\right) \\ &= \frac{3}{2}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\right] = \frac{3}{2}\sin 2x. \end{aligned}$$

20. $c_{n+2} = \dfrac{2^2 c_n}{(n+1)(n+2)}$ for $n \geq 0$, so $c_{2k} = \dfrac{2^{2k} c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{2^{2k} c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1 + \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} + \frac{2^6x^6}{6!} + \cdots\right) + c_1\left(x + \frac{2^2x^3}{3!} + \frac{2^4x^5}{5!} + \frac{2^6x^7}{7!} + \cdots\right)$$

$c_0 = y(0) = 2$ and $c_1 = y'(0) = 0$, so

$$y(x) = 2\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \cdots\right) = 2\cosh 2x.$$

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21. $c_{n+1} = \dfrac{2nc_n - c_{n-1}}{n(n+1)}$ for $n \geq 1$; with $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$, we obtain $c_2 = 1,\ c_3 = \dfrac{1}{2},\ c_4 = \dfrac{1}{6} = \dfrac{1}{3!},\ c_5 = \dfrac{1}{24} = \dfrac{1}{4!},\ c_6 = \dfrac{1}{120} = \dfrac{1}{5!}$. Evidently $c_n = \dfrac{1}{(n-1)!}$, so

$$y(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = xe^x.$$

22. $c_{n+1} = -\dfrac{nc_n - 2c_{n-1}}{n(n+1)}$ for $n \geq 1$; with $c_0 = y(0) = 1$ and $c_1 = y'(0) = -2$, we obtain $c_2 = 2,\ c_3 = -\dfrac{4}{3} = -\dfrac{2^3}{3!},\ c_4 = \dfrac{2}{3} = \dfrac{2^4}{4!},\ c_5 = -\dfrac{4}{15} = -\dfrac{2^5}{5!}$. Apparently $c_n = \pm\dfrac{2^n}{n!}$, so

$$y(x) = 1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \frac{(2x)^5}{5!} + \cdots = e^{-2x}.$$

23. $c_0 = c_1 = 0$ and the recursion relation

$$(n^2 - n + 1)c_n + (n - 1)c_{n-1} = 0$$

for $n \geq 2$ imply that $c_n = 0$ for $n \geq 0$. Thus any assumed power series solution $y = \Sigma c_n x^n$ must reduce to the trivial solution $y(x) \equiv 0$.

24. **(a)** The fact that $y(x) = (1 + x)^\alpha$ satisfies the differential equation $(1+x)y' = \alpha y$ follows immediately from the fact that $y'(x) = \alpha(1+x)^{\alpha-1}$.

(b) When we substitute $y = \Sigma c_n x^n$ into the differential equation $(1+x)y' = \alpha y$ we get the recurrence formula

$$c_{n+1} = \frac{(\alpha - n)c_n}{n+1}. c_{n+1} = (\alpha - n)c_n/(n + 1).$$

Since $c_0 = 1$ because of the initial condition $y(0) = 1$, the binomial series (Equation (12) in the text) follows.

(c) The function $(1 + x)^\alpha$ and the binomial series must agree on $(-1, 1)$ because of the uniqueness of solutions of linear initial value problems.

25. Substitution of $\sum_{n=0}^{\infty} c_n x^n$ into the differential equation $y'' = y' + y$ leads routinely — via shifts of summation to exhibit x^n-terms throughout — to the recurrence formula

$$(n+2)(n+1)c_{n+2} = (n+1)c_{n+1} + c_n,$$

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and the given initial conditions yield $c_0 = 0 = F_0$ and $c_1 = 1 = F_1$. But instead of proceeding immediately to calculate explicit values of further coefficients, let us first multiply the recurrence relation by $n!$. This trick provides the relation

$$(n+2)!c_{n+2} = (n+1)!c_{n+1} + n!c_n,$$

that is, the Fibonacci-defining relation $F_{n+2} = F_{n+1} + F_n$ where $F_n = n!c_n$, so we see that $c_n = F_n / n!$ as desired.

26. This problem is pretty fully outlined in the textbook. The only hard part is squaring the power series:

$$\begin{aligned}&\left(1 + c_3x^3 + c_5x^5 + c_7x^7 + c_9x^9 + c_{11}x^{11} + \cdots\right)^2 \\ &\quad = x^2 + 2c_3x^4 + \left(c_3^2 + 2c_5\right)x^6 + \left(2c_3c_5 + 2c_7\right)x^8 + \\ &\qquad \left(c_5^2 + 2c_3c_7 + 2c_9\right)x^{10} + \left(2c_5c_7 + 2c_3c_9 + 2c_{11}\right)x^{12} + \cdots\end{aligned}$$

27. **(b)** The roots of the characteristic equation $r^3 = 1$ are $r_1 = 1$, $r_2 = \alpha = (-1 + i\sqrt{3})/2$, and $r_3 = \beta = (-1 - i\sqrt{3})/2$. Then the general solution is

$$y(x) = Ae^x + Be^{\alpha x} + Ce^{\beta x}. \qquad (*)$$

Imposing the initial conditions, we get the equations

$$\begin{aligned} A + B + C &= 1 \\ A + \alpha B + \beta C &= 1 \\ A + \alpha^2 B + \beta^2 C &= -1. \end{aligned}$$

The solution of this system is $A = 1/3$, $B = (1 - i\sqrt{3})/3$, $C = (1 + i\sqrt{3})/3$. Substitution of these coefficients in (*) and use of Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$ finally yields the desired result.

SECTION 8.2

SERIES SOLUTIONS NEAR ORDINARY POINTS

Instead of deriving in detail the recurrence relations and solution series for Problems 1 through 15, we indicate where some of these problems and answers originally came from. Each of the differential equations in Problems 1–10 is of the form

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$$(Ax^2 + B)y'' + Cxy' + Dy = 0$$

with selected values of the constants A, B, C, D. When we substitute $y = \Sigma c_n x^n$, shift indices where appropriate, and collect coefficients, we get

$$\sum_{n=0}^{\infty}\left[An(n-1)c_n + B(n+1)(n+2)c_{n+2} + Cnc_n + Dc_n\right]x^n = 0.$$

Thus the recurrence relation is

$$c_{n+2} = -\frac{An^2 + (C-A)n + D}{B(n+1)(n+2)}c_n \qquad \text{for } n \geq 0.$$

It yields a solution of the form

$$y = c_0\, y_{\text{even}} + c_1\, y_{\text{odd}}$$

where y_{even} and y_{odd} denote series with terms of even and odd degrees, respectively. The even-degree series $c_0 + c_2x^2 + c_4x^4 + \cdots$ converges (by the ratio test) provided that

$$\lim_{n\to\infty}\left|\frac{c_{n+2}x^{n+2}}{c_nx^n}\right| = \left|\frac{Ax^2}{B}\right| < 1.$$

Hence its radius of convergence is at least $\rho = \sqrt{|B/A|}$, as is that of the odd-degree series $c_1x + c_3x^3 + c_5x^4 + \cdots$. (See Problem 6 for an example in which the radius of convergence is, surprisingly, greater than $\sqrt{|B/A|}$.)

In Problems 1–15 we give first the recurrence relation and the radius of convergence, then the resulting power series solution.

1. $c_{n+2} = c_n;$ $\qquad \rho = 1;$ $\qquad c_0 = c_2 = c_4 = \cdots;$ $\qquad c_1 = c_3 = c_5 = \cdots$

$$y(x) = c_0\sum_{n=0}^{\infty}x^{2n} + c_1\sum_{n=0}^{\infty}x^{2n+1} = \frac{c_0 + c_1x}{1-x^2}$$

2. $c_{n+2} = -\frac{1}{2}c_n;$ $\qquad \rho = 2;$ $\qquad c_{2n} = \frac{(-1)^n c_0}{2^n};$ $\qquad c_{2n+1} = \frac{(-1)^n c_1}{2^n}$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n} + c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n}$$

3. $c_{n+2} = -\frac{c_n}{(n+2)};$ $\qquad \rho = \infty;$

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$$c_{2n} = \frac{(-1)^n c_0}{(2n)(2n-2)\cdots\cdots 4\cdot 2} = \frac{(-1)^n c_0}{n!2^n};$$

$$c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)(2n-1)\cdots\cdots 5\cdot 3} = \frac{(-1)^n c_1}{(2n+1)!!}$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!2^n} + c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!!}$$

4. $c_{n+2} = -\frac{n+4}{n+2}c_n; \qquad \rho = 1$

$$c_{2n} = \left(-\frac{2n+2}{2n}\right)\left(-\frac{2n}{2n-2}\right)\cdots\cdots\left(-\frac{6}{4}\right)\left(-\frac{4}{2}\right)c_0 = (-1)^n\frac{2n+2}{2}c_0 = (-1)^n(n+1)c_0$$

$$c_{2n} = \left(-\frac{2n+3}{2n+1}\right)\left(-\frac{2n+1}{2n-1}\right)\cdots\cdots\left(-\frac{7}{5}\right)\left(-\frac{5}{3}\right)c_0 = (-1)^n\frac{2n+3}{3}c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n} + \frac{1}{3}c_1\sum_{n=0}^{\infty}(-1)^n(2n+3)x^{2n+1}$$

5. $c_{n+2} = \frac{nc_n}{3(n+2)}; \qquad \rho = \sqrt{3}; \qquad c_2 = c_4 = c_6 = \cdots = 0$

$$c_{2n+1} = \frac{2n-1}{3(2n+1)}\cdot\frac{2n-3}{3(2n-1)}\cdots\cdots\frac{3}{3(5)}\cdot\frac{1}{3(3)}c_1 = \frac{c_1}{(2n+1)3^n}$$

$$y(x) = c_0 + c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)3^n}$$

6. $c_{n+2} = \frac{(n-3)(n-4)}{(n+1)(n+2)}c_n$

The factor $(n-3)$ in the numerator yields $c_5 = c_7 = c_9 = \cdots = 0,$ and the factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0.$ Hence y_{even} and y_{odd} are both polynomials with radius of convergence $\rho = \infty.$

$$y(x) = c_0(1+6x^2+x^4) + c_1(x+x^3)$$

7. $c_{n+2} = -\frac{(n-4)^2}{3(n+1)(n+2)}c_n; \qquad \rho \ge \sqrt{3}$

The factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0,$ so y_{even} is a 4th-degree polynomial.

We find first that $c_3 = -c_1/2$ and $c_5 = c_1/120$, and then for $n \ge 3$ that

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$$c_{2n+1} = \left(-\frac{(2n-5)^2}{3(2n)(2n+1)}\right)\left(-\frac{(2n-7)^2}{3(2n-2)(2n-1)}\right)\cdots\cdots\left(-\frac{1^2}{3(6)(7)}\right)c_5 =$$

$$= (-1)^{n-2}\frac{\left[(2n-5)!!\right]^2}{3^{n-2}(2n+1)(2n-1)\cdots\cdots 7\cdot 6}\cdot\frac{c_1}{120} = 9\cdot(-1)^n\frac{\left[(2n-5)!!\right]^2}{3^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-\frac{8}{3}x^2+\frac{8}{27}x^4\right)+c_1\left[x-\frac{1}{2}x^3+\frac{1}{120}x^5+9\sum_{n=3}^{\infty}\frac{[(2n-5)!!]^2(-1)^n}{(2n+1)!\,3^n}x^{2n+1}\right]$$

8. $$c_{n+2} = \frac{(n-4)(n+4)}{2(n+1)(n+2)}c_n; \qquad \rho\ge\sqrt{2}$$

We find first that $c_3=-5c_1/4$ and $c_5=7c_1/32$, and then for $n\ge 3$ that

$$c_{2n+1} = \left(\frac{(2n-5)(2n+3)}{2(2n)(2n+1)}\right)\left(\frac{(2n-7)(2n+1)}{2(2n-2)(2n-1)}\right)\cdots\cdots\left(\frac{1\cdot 9}{2(6)(7)}\right)c_5 =$$

$$= \frac{(2n-5)!!(2n+3)(2n+1)\cdots\cdots 9}{2^{n-2}(2n+1)(2n)\cdots\cdots 7\cdot 6}\cdot\frac{7c_1}{32} = 4\cdot\frac{5!}{7\cdot 5\cdot 3}\cdot\frac{7}{32}\frac{(2n-5)!!(2n+3)!!}{2^n(2n+1)!}c_1$$

$$c_{2n+1} = \frac{(2n-5)!!(2n+3)!!}{2^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-4x^2+2x^4\right)+c_1\left[x-\frac{5}{4}x^3+\frac{7}{32}x^5+\sum_{n=3}^{\infty}\frac{(2n-5)!!(2n+3)!!}{(2n+1)!\,2^n}x^{2n+1}\right]$$

9. $$c_{n+2} = \frac{(n+3)(n+4)}{(n+1)(n+2)}c_n; \qquad \rho=1$$

$$c_{2n} = \frac{(2n+1)(2n+2)}{(2n-1)(2n)}\cdot\frac{(2n-1)(2n)}{(2n-3)(2n-2)}\cdots\cdots\frac{3\cdot 4}{1\cdot 2}c_0 = (n+1)(2n+1)c_0$$

$$c_{2n+1} = \frac{(2n+2)(2n+3)}{(2n)(2n+1)}\cdot\frac{(2n)(2n+1)}{(2n-2)(2n-1)}\cdots\cdots\frac{4\cdot 5}{2\cdot 3}c_1 = \frac{1}{3}(n+1)(2n+3)c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}c_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}$$

10. $$c_{n+2} = -\frac{(n-4)}{3(n+1)(n+2)}c_n; \qquad \rho=\infty$$

The factor $(n-4)$ yields $c_6=c_8=c_{10}=\cdots=0$, so y_{even} is a 4th-degree polynomial. We find first that $c_3=c_1/6$ and $c_5=c_1/360$, and then for $n\ge 3$ that

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$$c_{2n+1} = \frac{-(2n-5)}{3(2n+1)(2n)} \cdot \frac{-(2n-3)}{3(2n-1)(2n-2)} \cdot \cdots \cdot \frac{-1}{3(7)(6)} c_5$$

$$= \frac{(2n-5)!!(-1)^{n-2}}{3^{n-2}(2n+1)(2n)\cdot\cdots\cdot(7)(6)} \cdot \frac{c_1}{360} =$$

$$= \frac{3^2 \cdot 5!}{360} \cdot \frac{(2n-5)!!(-1)^n}{3^n(2n+1)(2n)\cdot\cdots\cdot(7)(6)\cdot 5!} \cdot c_1 = 3 \cdot \frac{(2n-5)!!(-1)^n}{3^n(2n+1)!} c_1$$

$$y(x) = c_0\left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) + c_1\left[x + \frac{1}{6}x^3 + \frac{1}{360}x^5 + 3\sum_{n=3}^{\infty}\frac{(2n-5)!!(-1)^n}{(2n+1)!\,3^n}x^{2n+1}\right]$$

11. $c_{n+2} = \dfrac{2(n-5)}{5(n+1)(n+2)}c_n; \qquad \rho = \infty$

The factor $(n-5)$ yields $c_7 = c_9 = c_{11} = \cdots = 0$, so y_{odd} is a 5th-degree polynomial. We find first that $c_2 = -c_1$, $c_4 = c_0/10$ and $c_6 = c_0/750$, and then for $n \geq 4$ that

$$c_{2n} = \frac{2(2n-7)}{5(2n)(2n-1)} \cdot \frac{2(2n-5)}{5(2n-2)(2n-3)} \cdot \cdots \cdot \frac{2(1)}{5(8)(7)} c_6$$

$$= \frac{2^{n-3}(2n-7)!!}{5^{n-3}(2n)(2n-1)\cdot\cdots\cdot(8)(7)} \cdot \frac{c_0}{750} =$$

$$= \frac{5^3 \cdot 6!}{2^3 \cdot 750} \cdot \frac{2^n(2n-7)!!}{5^n(2n)(2n)\cdot\cdots\cdot(8)(7)\cdot 6!} \cdot c_1 = 15 \cdot \frac{2^n(2n-7)!!}{5^n(2n)!} c_0$$

$$y(x) = c_1\left(x - \frac{4x^3}{15} + \frac{4x^5}{375}\right) + c_0\left[1 - x^2 + \frac{x^4}{10} + \frac{x^6}{750} + 15\sum_{n=4}^{\infty}\frac{(2n-7)!!\,2^n}{(2n)!\,5^n}x^{2n}\right]$$

12. $c_{n+3} = \dfrac{c_n}{n+2}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0$, so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0\left[1 + \sum_{n=1}^{\infty}\frac{x^{3n}}{2 \cdot 5 \cdot \cdots \cdot (3n-1)}\right] + c_1\sum_{n=0}^{\infty}\frac{x^{3n+1}}{n!\,3^n}$$

13. $c_{n+3} = -\dfrac{c_n}{n+3}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0$, so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!\,3^n} + c_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{1 \cdot 4 \cdot \cdots \cdot (3n+1)}$$

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14. $c_{n+3} = -\dfrac{c_n}{(n+2)(n+3)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0$, so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also. Then

$$c_{3n} = \frac{-1}{(3n)(3n-1)} \cdot \frac{-1}{(3n-3)(3n-4)} \cdot \cdots \cdot \frac{-1}{3 \cdot 2} c_0 = \frac{(-1)^n c_0}{3^n n! \cdot (3n-1)(3n-4) \cdot \cdots \cdot 5 \cdot 2},$$

$$c_{3n+1} = \frac{-1}{(3n+1)(3n)} \cdot \frac{-1}{(3n-2)(3n-3)} \cdot \cdots \cdot \frac{-1}{4 \cdot 3} c_1 = \frac{(-1)^n c_1}{3^n n! \cdot (3n+1)(3n-2) \cdot \cdots \cdot 4 \cdot 1}.$$

$$y(x) = c_0\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{3n}}{3^n n! \cdot 2 \cdot 5 \cdot \cdots \cdot (3n-1)}\right] + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3^n n! \cdot 1 \cdot 4 \cdot \cdots \cdot (3n+1)}$$

15. $c_{n+4} = -\dfrac{c_n}{(n+3)(n+4)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_3 = 0$, so the recurrence relation yields $c_6 = c_{10} = \cdots = 0$ and $c_7 = c_{11} = \cdots = 0$ also. Then

$$c_{4n} = \frac{-1}{(4n)(4n-1)} \cdot \frac{-1}{(4n-4)(4n-5)} \cdot \cdots \cdot \frac{-1}{4 \cdot 3} c_0 = \frac{(-1)^n c_0}{4^n n! \cdot (4n-1)(4n-5) \cdot \cdots \cdot 5 \cdot 3},$$

$$c_{3n+1} = \frac{-1}{(4n+1)(4n)} \cdot \frac{-1}{(4n-3)(4n-4)} \cdot \cdots \cdot \frac{-1}{5 \cdot 4} c_1 = \frac{(-1)^n c_1}{4^n n! \cdot (4n+1)(4n-3) \cdot \cdots \cdot 9 \cdot 5}.$$

$$y(x) = c_0\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{4^n n! \cdot 3 \cdot 7 \cdot \cdots \cdot (4n-1)}\right] + c_1\left[x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+1}}{4^n n! \cdot 5 \cdot 9 \cdot \cdots \cdot (4n+1)}\right]$$

16. The recurrence relation is $c_{n+2} = -\dfrac{n-1}{n+1} c_n$ for $n \geq 1$. The factor $(n-1)$ in the numerator yields $c_3 = c_5 = c_7 = \cdots = 0$. When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_0$, and then the recurrence relation gives

$$c_{2n} = -\frac{2n-3}{2n-1} \cdot -\frac{2n-5}{2n-3} \cdot \cdots \cdot -\frac{3}{5} \cdot -\frac{1}{3} c_2 = \frac{(-1)^{n-1}}{2n-1} c_0.$$

Hence

$$y(x) = c_1 x + c_0\left(1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \cdots\right)$$

$$= c_1 x + c_0 + c_0 x\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = c_1 x + c_0\left(1 + x \tan^{-1} x\right).$$

With $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$ we obtain the particular solution $y(x) = x$.

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17. The recurrence relation

$$c_{n+2} = -\frac{(n-2)c_n}{(n+1)(n+2)}$$

yields $c_2 = c_0 = y(0) = 1$ and $c_4 = c_6 = \cdots = 0$. Because $c_1 = y'(0) = 0$, it follows also that $c_1 = c_3 = c_5 = \cdots = 0$. Thus the desired particular solution is $y(x) = 1 + x^2$.

18. The substitution $t = x - 1$ yields $y'' + ty' + y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{c_n}{n+2}.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = \infty$. The initial conditions give $c_0 = 2$ and $c_1 = 0$, so $c_{\text{odd}} = 0$ and it follows that

$$y = 2\left(1 - \frac{t^2}{2} + \frac{t^4}{2\cdot 4} - \frac{t^6}{2\cdot 4\cdot 6} + \cdots\right),$$

$$y(x) = 2\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^4}{2\cdot 4} - \frac{(x-1)^6}{2\cdot 4\cdot 6} + \cdots\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^n (x-1)^{2n}}{n!2^n}.$$

19. The substitution $t = x - 1$ yields $(1 - t^2)y'' - 6ty' - 4y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{n+4}{n+2}c_n.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = 1$, and therefore converges if $-1 < t < 1$. The initial conditions give $c_0 = 0$ and $c_1 = 1$, so $c_{\text{even}} = 0$ and

$$c_{2n+1} = \frac{2n+3}{2n+1}\cdot\frac{2n+1}{2n-1}\cdot\cdots\cdot\frac{7}{5}\cdot\frac{5}{3}c_1 = \frac{2n+3}{3}.$$

Thus

$$y = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)t^{2n+1} = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)(x-1)^{2n+1},$$

and the x-series converges if $0 < x < 2$.

20. The substitution $t = x - 3$ yields $(t^2 + 1)y'' - 4ty' + 6y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}c_n$$

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for $n \geq 0$. The initial conditions give $c_0 = 2$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = -6$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2 - 6t^2 = 2 - 6(x-3)^2.$$

21. The substitution $t = x + 2$ yields $(4t^2 + 1)y'' = 8y$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{4(n-2)}{(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 1$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = 4$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2 + 4t^2 = 1 + 4(x+2)^2.$$

22. The substitution $t = x + 3$ yields $(t^2 - 9)y'' + 3ty' - 3y = 0$, with primes now denoting differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{(n+3)(n-1)}{9(n+1)(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 0$ and $c_1 = 2$. It follows that $c_{\text{even}} = 0$ and $c_3 = c_5 = \cdots = 0$, so

$$y = 2t = 2x + 6.$$

In Problems 23–26 we first derive the recurrence relation, and then calculate the solution series $y_1(x)$ with $c_0 = 1$ and $c_1 = 0$ as well as the solution series $y_2(x)$ with $c_0 = 0$ and $c_1 = 1$.

23. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + \sum_{n=1}^{\infty}\left[c_{n-1} + c_n + (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = -\frac{1}{2}c_0, \qquad c_{n+2} = -\frac{c_{n-1} + c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 1.$$

$$y_1(x) = 1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots; \qquad y_2(x) = x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{120} + \cdots$$

24. Substitution of $y = \Sigma c_n x^n$ yields

$$-2c_2 + \sum_{n=1}^{\infty}\left[2c_{n-1} + n(n+1)c_n - (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

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so

$$c_2 = 0, \qquad c_{n+2} = \frac{2c_{n-1} + n(n+1)c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 1.$$

$$y_1(x) = 1 + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^6}{45} + \cdots; \qquad y_2(x) = x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{5} + \cdots$$

25. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + \sum_{n=2}^{\infty} \left[c_{n-2} + (n-1)c_{n-1} + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = c_3 = 0, \qquad c_{n+2} = -\frac{c_{n-2} + (n-1)c_{n-1}}{(n+1)(n+2)} \quad \text{for} \quad n \geq 2.$$

$$y_1(x) = 1 - \frac{x^4}{12} + \frac{x^7}{126} + \frac{x^8}{672} + \cdots; \qquad y_2(x) = x - \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^7}{126} + \cdots$$

26. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + 12c_4 x^2 + (2c_2 + 20c_5)x^3 + \sum_{n=4}^{\infty} \left[c_{n-4} + (n-1)(n-2)c_{n-1} + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = c_3 = c_4 = c_5 = 0, \qquad c_{n+2} = -\frac{c_{n-4} + (n-1)(n-2)c_{n-1}}{(n+1)(n+2)} \quad \text{for} \quad n \geq 4.$$

$$y_1(x) = 1 - \frac{x^6}{30} + \frac{x^9}{72} - \frac{29x^{12}}{3960} + \cdots; \qquad y_2(x) = x - \frac{x^7}{42} + \frac{x^{10}}{90} - \frac{41x^{13}}{6552} + \cdots$$

27. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + (2c_1 + 6c_3)x + \sum_{n=2}^{\infty} \left[2c_{n-2} + (n+1)c_n + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = -\frac{c_0}{2}, \quad c_3 = -\frac{c_1}{3}, \qquad c_{n+2} = -\frac{2c_{n-2} + (n+1)c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 2.$$

With $c_0 = y(0) = 1$ and $c_1 = y'(0) = -1$, we obtain

$$y(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} - \frac{13x^7}{630} - \frac{143x^8}{40320} + \frac{31x^9}{22680} + \cdots.$$

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Finally, $x = 0.5$ gives

$$\begin{aligned} y(0.5) &= 1 - 0.5 - 0.125 + 0.041667 - 0.002604 + 0.001042 \\ &\quad + 0.000629 - 0.000161 - 0.000014 + 0.000003 + \cdots \\ y(0.5) &\approx 0.415562 \approx 0.4156. \end{aligned}$$

28. When we substitute $y = \Sigma c_n x^n$ and $e^{-x} = \sum (-1)^n x^n / n!$ and then collect coefficients of the terms involving $1,\ x,\ x^2$, and x^3, we find that

$$c_2 = -\frac{c_0}{2},\quad c_3 = \frac{c_0 - c_1}{6},\quad c_4 = \frac{c_1}{12},\quad c_5 = -\frac{3c_0 + 2c_1}{120}.$$

With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{40} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{60} + \cdots.$$

29. When we substitute $y = \Sigma c_n x^n$ and $\cos x = \sum (-1)^n x^{2n} / (2n)!$ and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^6$, we obtain the equations

$$c_0 + 2c_2 = 0,\quad c_1 + 6c_3 = 0,\quad 12c_4 = 0,\quad -2c_3 + 20c_5 = 0,$$

$$\frac{1}{12}c_2 - 5c_4 + 30c_6 = 0,\quad \frac{1}{4}c_3 - 9c_5 + 42c_6 = 0,$$

$$-\frac{1}{360}c_2 + \frac{1}{2}c_4 - 14c_6 + 56c_8 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_8$ in turn. With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^6}{720} + \frac{13x^8}{40320} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} - \frac{x^5}{60} - \frac{13x^7}{5040} + \cdots.$$

30. When we substitute $y = \Sigma c_n x^n$ and $\sin x = \Sigma (-1)^n x^{2n+1} / (2n+1)!$, and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^5$, we obtain the equations

$$c_0 + c_1 + 2c_2 = 0,\quad c_1 + 2c_2 + 6c_3 = 0,\quad -\frac{c_1}{6} + c_2 + 3c_3 + 12c_4 = 0,$$

$$-\frac{c_2}{3} + c_3 + 4c_4 + 20c_5 = 0,\quad \frac{c_1}{120} - \frac{c_3}{2} + c_4 + 5c_5 + 30c_6 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_6$ in turn. With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

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$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{60} + \frac{x^6}{180} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^2}{2} + \frac{x^4}{18} - \frac{7x^5}{360} + \frac{x^6}{900} + \cdots.$$

33. Substitution of $y = \Sigma c_n x^n$ in Hermite's equation leads in the usual way to the recurrence formula

$$c_{n+2} = -\frac{2(\alpha - n)c_n}{(n+1)(n+2)}.$$

Starting with $c_0 = 1$, this formula yields

$$c_2 = -\frac{2\alpha}{2!}, \quad c_4 = +\frac{2^2\alpha(\alpha-2)}{4!}, \quad c_6 = -\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}, \ \ldots.$$

Starting with $c_1 = 1$, it yields

$$c_3 = -\frac{2(\alpha-1)}{3!}, \quad c_5 = +\frac{2^2(\alpha-1)(\alpha-3)}{5!}, \quad c_7 = -\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}, \ \ldots.$$

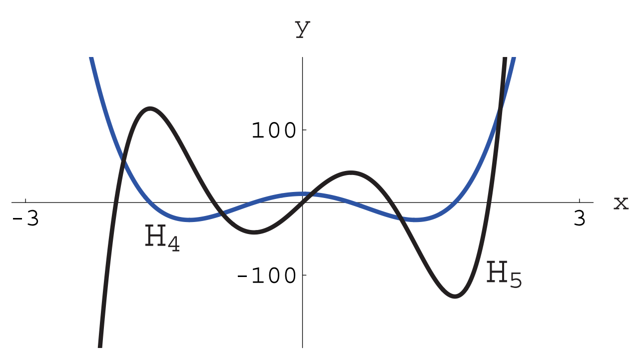
This gives the desired even-term and odd-term series y_1 and y_2. If α is an integer, then obviously one series or the other has only finitely many non-zero terms. For instance, with $\alpha = 4$ we get

$$y_1(x) = 1 - \frac{2 \cdot 4}{2}x^2 + \frac{2^2 \cdot 4 \cdot 2}{24}x^4 = 1 - 4x^2 + \frac{4}{3}x^4 = \frac{1}{12}\left(16x^4 - 48x^2 + 12\right),$$

and with $\alpha = 5$ we get

$$y_2(x) = x - \frac{2 \cdot 4}{6}x^3 + \frac{2^2 \cdot 4 \cdot 2}{120}x^5 = x - \frac{4}{3}x^3 + \frac{4}{15}x^5 = \frac{1}{120}\left(32x^5 - 160x^3 + 120\right).$$

The figure below shows the interlaced zeros of the 4th and 5th Hermite polynomials.



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34. Substitution of $y = \Sigma c_n x^n$ in the Airy equation leads upon shift of index and collection of terms to

$$2c_2 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)c_{n+2} - c_{n-1}\right]x^n = 0.$$

The identity principle then gives $c_2 = 0$ and the recurrence formula

$$c_{n+3} = \frac{c_n}{(n+2)(n+3)}.$$

Because of the "3-step" in indices, it follows that $c_2 = c_5 = c_8 = c_{11} = \cdots = 0$. Starting with $c_0 = 1$, we calculate

$$c_3 = \frac{1}{2\cdot 3} = \frac{1}{3!} =, \quad c_6 = \frac{1}{3!\cdot 5\cdot 6} = \frac{1\cdot 4}{6!}, \quad c_9 = \frac{1\cdot 4}{6!\cdot 8\cdot 9} = \frac{1\cdot 4\cdot 7}{9!}, \ \ldots .$$

Starting with $c_1 = 1$, we calculate

$$c_4 = \frac{1}{3\cdot 4} = \frac{2}{4!}, \quad c_7 = \frac{2}{4!\cdot 6\cdot 7} = \frac{2\cdot 5}{7!}, \quad c_{10} = \frac{2\cdot 5}{7!\cdot 9\cdot 10} = \frac{2\cdot 5\cdot 8}{10!}, \ \ldots .$$

Evidently we are building up the coefficients

$$c_{3k} = \frac{1\cdot 4\cdots(3k-2)}{(3k)!} \qquad \text{and} \qquad c_{3k+1} = \frac{2\cdot 5\cdots(3k-1)}{(3k+1)!}$$

that appear in the desired series for $y_1(x)$ and $y_2(x)$. Finally, the *Mathematica* commands

```
A[1] = 1/6; A[k_] := A[k - 1]/(3 k*(3 k - 1));
B[1] = 1/12; B[k_] := B[k - 1]/(3 k*(3 k + 1));
n = 40;
y1 = 1 + Sum[A[k]*x^(3 k), {k, 1, n}];
y2 = x + Sum[B[k]*x^(3 k + 1), {k, 1, n}];
yA = y1/(3^(2/3)*Gamma[2/3])  - y2/(3^(1/3)*Gamma[1/3]);
yB = y1/(3^(1/6)*Gamma[2/3]) + y2/(3^(-1/6)*Gamma[1/3]);
Plot[{yA, yB}, {x, -13.5, 3}, PlotRange -> {-0.75, 1.5}]
```

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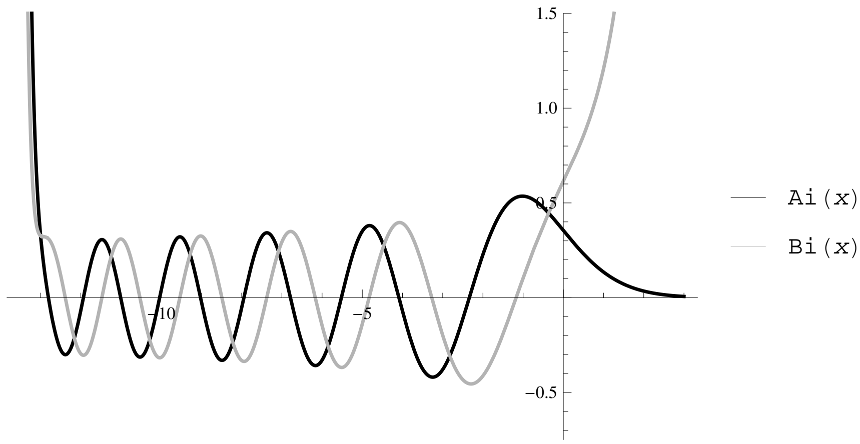

produce the figure above. But with $n = 50$ (instead of $n = 40$) terms we get a figure that is visually indistinguishable from Figure 8.2.3 in the textbook.

35. **(a)** If

$$y_0 = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^{3n} n!} x^{2n} = 1 + \sum_{n=1}^{\infty} a_n z^n$$

where $a_n = \dfrac{(2n-1)!!}{2^{3n} n!}$, then the radius of convergence of the series in $z = x^2$ is

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(2n-1)!!/2^{3n} n!}{(2n+1)!!/2^{3n+3}(n+1)!} = \lim_{n\to\infty} \frac{2^3(n+1)}{2n+1} = 4.$$

Thus the series in z converges if $-4 < z = x^2 < 4$, so the series $y_0(x)$ converges if $-2 < x < 2$, and thus has radius of convergence equal to 2.

(b) If

$$y_1 = x\left(1 + \sum_{n=1}^{\infty} \frac{n!}{2^n (2n+1)!!} x^{2n}\right) = x\left(1 + \sum_{n=1}^{\infty} b_n z^n\right)$$

where $b_n = \dfrac{n!}{2^n (2n+1)!!}$, then the radius of convergence of the series in z is

$$\rho = \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| = \lim_{n\to\infty} \frac{n!/2^n (2n+1)!!}{(n+1)!/2^{n+1}(2n+3)!!} = \lim_{n\to\infty} \frac{2(2n+3)}{n+1} = 4.$$

Hence it follows as in part (a) that the series $y_1(x)$ has radius of convergence equal to 2.

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SECTION 8.3

REGULAR SINGULAR POINTS

1. Upon division of the given differential equation by x we see that $P(x) = 1 - x^2$ and $Q(x) = (\sin x)/x$. Because both are analytic at $x = 0$ — in particular, $(\sin x)/x \to 1$ as $x \to 0$ because

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

— it follows that $x = 0$ is an ordinary point.

2. Division of the differential equation by x yields

$$y'' + xy' + \frac{e^x - 1}{x}y = 0.$$

Because the function

$$\frac{e^x - 1}{x} = \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - 1\right) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

is analytic at the origin, we see that $x = 0$ is an ordinary point.

3. When we rewrite the given equation in the standard form of Equation (3) in this section, we see that $p(x) = (\cos x)/x$ and $q(x) = x$. Because $(\cos x)/x \to \infty$ as $x \to 0$ it follows that $p(x)$ is not analytic at $x = 0$, so $x = 0$ is an irregular singular point.

4. When we rewrite the given equation in the standard form of Equation (3), we have $p(x) = 2/3$ and $q(x) = (1 - x^2)/3x$. Since $q(x)$ is not analytic at the origin, $x = 0$ is an irregular singular point.

5. In the standard form of Equation (3) we have $p(x) = 2/(1 + x)$ and $q(x) = 3x^2/(1 + x)$. Both are analytic $x = 0$, so $x = 0$ is a regular singular point. The indicial equation is

$$r(r - 1) + 2r = r^2 + r = r(r + 1) = 0,$$

so the exponents are $r_1 = 0$ and $r_2 = -1$.

6. In the standard form of Equation (3) we have $p(x) = 2/(1 - x^2)$ and $q(x) = -2/(1 - x^2)$, so $x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = -2$. The indicial equation is $r^2 + r - 2 = 0$, so the exponents are $r = -2, 1$.

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7. In the standard form of Equation (3) we have $p(x) = (6\sin x)/x$ and $q(x) = 6$, so $x = 0$ is a regular singular point with $p_0 = q_0 = 6$. The indicial equation is $r^2 + 5r + 6 = 0$, so the exponents are $r_1 = -2$ and $r_2 = -3$.

8. In the standard form of Equation (3) we have $p(x) = 21/(6+2x)$ and $q(x) = 9(x^2-1)/(6+2x)$, so $x = 0$ is a regular singular point with $p_0 = 7/2$ and $q_0 = -3/2$. The indicial equation simplifies to $2r^2 + 5r - 3 = 0$, so the exponents are $r = -3, 1/2$.

9. The only singular point of the differential equation $y'' + \dfrac{x}{1-x}y' + \dfrac{x^2}{1-x}y = 0$ is $x = 1$. Upon substituting $t = x-1$, $x = t+1$ we get the transformed equation $y'' - \dfrac{t+1}{t}y' - \dfrac{(t+1)^2}{t}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = -(1+t)$ and $q(t) = -t(1+t)^2$. Both these functions are analytic, so it follows that $x = 1$ is a regular singular point of the original equation.

10. The only singular point of the differential equation $y'' + \dfrac{2}{x-1}y' + \dfrac{1}{(x-1)^2}y = 0$ is $x = 1$. Upon substituting $t = x-1$, $x = t+1$ we get the transformed equation $y'' + \dfrac{2}{t}y' + \dfrac{1}{t^2}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 2$ and $q(t) \equiv 1$. Both these functions are analytic, so it follows that $x = 1$ is a regular singular point of the original equation.

11. The only singular points of the differential equation $y'' - \dfrac{2x}{1-x^2}y' + \dfrac{12}{1-x^2}y = 0$ are $x = +1$ and $x = -1$.

$\boldsymbol{x = +1}$: Upon substituting $t = x-1$, $x = t+1$ we get the transformed equation $y'' + \dfrac{2(t+1)}{t(t+2)}y' - \dfrac{12}{t(t+2)}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{2(t+1)}{t+2}$ and $q(t) = -\dfrac{12t}{t+2}$. Both these functions are analytic at $t = 0$, so it follows that $x = +1$ is a regular singular point of the original equation.

$\boldsymbol{x = -1}$: Upon substituting $t = x+1$, $x = t-1$ we get the transformed equation $y'' + \dfrac{2(t-1)}{t(t-2)}y' - \dfrac{12}{t(t-2)}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{2(t-1)}{t-2}$ and $q(t) = -\dfrac{12t}{t-2}$.

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Both these functions are analytic at $t = 0$, so it follows that $x = -1$ is a regular singular point of the original equation.

12. The only singular point of the differential equation $y'' + \dfrac{3}{x-2}y' + \dfrac{x^3}{(x-2)^3}y = 0$ is $x = 2$. Upon substituting $t = x - 2$, $x = t + 2$ we get the transformed equation $y'' + \dfrac{3}{t}y' + \dfrac{(t+2)^3}{t^3}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 3$ and $q(t) = \dfrac{(t+2)^3}{t}$. Because q is *not* analytic at $t = 0$, it follows that $x = 2$ is an irregular singular point of the original equation.

13. The only singular points of the differential equation $y'' + \dfrac{1}{x+2}y' + \dfrac{1}{x-2}y = 0$ are $x = +2$ and $x = -2$.

***x* = +2**: Upon substituting $t = x - 2$, $x = t + 2$ we get the transformed equation $y'' + \dfrac{1}{t+4}y' + \dfrac{1}{t}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{t}{t+4}$ and $q(t) = t$. Both these functions are analytic at $t = 0$, so it follows that $x = +2$ is a regular singular point of the original equation.

***x* = –2**: Upon substituting $t = x + 2$, $x = t - 2$ we get the transformed equation $y'' + \dfrac{1}{t}y' + \dfrac{1}{t-4}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 1$ and $q(t) = \dfrac{t^2}{t-4}$. Both these functions are analytic at $t = 0$, so it follows that $x = -2$ is a regular singular point of the original equation.

14. The only singular points of the differential equation $y'' + \dfrac{x^2+9}{(x^2-9)^2}y' + \dfrac{x^2+4}{(x^2-9)^2}y = 0$ are $x = +3$ and $x = -3$.

***x* = +3**: Upon substituting $t = x - 3$, $x = t + 3$ we get the transformed equation $y'' + \dfrac{t^2+6t+13}{t^2(t^2+6)^2}y' + \dfrac{t^2+6t+18}{t^2(t^2+6)^2}y = 0$, where primes now denote differentiation with

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respect to t. Because $p(t)=\dfrac{t^2+6t+13}{t(t^2+6)^2}$ is *not* analytic at $t=0$, it follows that $x = 3$ is an irregular singular point of the original equation.

***x* = –3**: Upon substituting $t = x+3,\ x = t-3$ we get the transformed equation $y''+\dfrac{t^2-6t+13}{t^2(t^2-6)^2}y'+\dfrac{t^2-6t+18}{t^2(t^2-6)^2}y = 0,$ where primes now denote differentiation with respect to t. Because $p(t)=\dfrac{t^2-6t+13}{t(t^2-6)^2}$ is *not* analytic at $t=0$, it follows that $x = -3$ is an irregular singular point of the original equation.

15. The only singular point of the differential equation $y''-\dfrac{x^2-4}{(x-2)^2}y'+\dfrac{x+2}{(x-2)^2}y = 0$ is $x = 2$. Upon substituting $t = x-2,\ x = t+2$ we get the transformed equation $y''-\dfrac{t+4}{t}y'+\dfrac{t+4}{t^2}y = 0,$ where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = -(t+4)$ and $q(t) = t+4$. Both these functions are analytic, so it follows that $x = 2$ is a regular singular point of the original equation.

16. The only singular points of the differential equation $y''+\dfrac{3x+2}{x^3(1-x)}y'+\dfrac{1}{x^2(1-x)}y = 0$ are $x = 0$ and $x = 1$.

***x* = 0**: In the standard form of Equation (3) we have $p(x) = \dfrac{3x+2}{x^2(1-x)}$ and $q(x) = \dfrac{1}{1-x}$. Since p is not analytic at $x=0,$ it follows that $x=0$ is an irregular singular point.

***x* = 1**: Upon substituting $t = x-1,\ x = t+1$ we get the transformed equation $y''-\dfrac{3t+5}{(t+1)^3}y'-\dfrac{t}{(t+1)^2}y = 0,$ where primes now denote differentiation with respect to t. Both $p(t) \equiv -\dfrac{t(3t+5)}{(t+1)^3}$ and $q(t) = -\dfrac{t^3}{(t+1)^2}$ are analytic at $t=0,$ so it follows that $x = 1$ is a regular singular point of the original equation.

Each of the differential equations in Problems 17–20 is of the form

$$Axy'' + By' + Cy = 0$$

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with indicial equation $Ar^2 + (B - A)r = 0$. Substitution of $y = \Sigma c_n x^{n+r}$ into the differential equation yields the recurrence relation

$$c_n = -\frac{C\,c_{n-1}}{A(n+r)^2 + (B-A)(n+r)}$$

for $n \ge 1$. In these problems the exponents $r_1 = 0$ and $r_2 = (A - B)/A$ do *not* differ by an integer, so this recurrence relation yields two linearly independent Frobenius series solutions when we apply it separately with $r = r_1$ and with $r = r_2$.

17. With exponent $r_1 = 0$: $c_n = -\dfrac{c_{n-1}}{4n^2 - 2n}$

$$y_1(x) = x^0\left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots\right) = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{x}\right)^{2n}}{(2n)!} = \cos\sqrt{x}$$

With exponent $r_2 = \dfrac{1}{2}$: $c_n = -\dfrac{c_{n-1}}{4n^2 + 2n}$

$$y_2(x) = x^{1/2}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \cdots\right) = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{x}\right)^{2n+1}}{(2n+1)!} = \sin\sqrt{x}$$

18. With exponent $r_1 = 0$: $c_n = \dfrac{c_{n-1}}{2n^2 + n}$

$$y_1(x) = x^0\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \cdots\right) = \sum_{n=0}^{\infty}\frac{x^n}{n!(2n+1)!!}$$

With exponent $r_2 = -\dfrac{1}{2}$: $c_n = \dfrac{c_{n-1}}{2n^2 - n}$

$$y_2(x) = x^{-1/2}\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \cdots\right) = \frac{1}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty}\frac{x^n}{n!(2n-1)!!}\right]$$

19. With exponent $r_1 = 0$: $c_n = \dfrac{c_{n-1}}{2n^2 - 3n}$

$$y_1(x) = x^0\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \frac{x^4}{360} - \cdots\right) = 1 - x - \sum_{n=2}^{\infty}\frac{x^n}{n!(2n-3)!!}$$

With exponent $r_2 = \dfrac{3}{2}$: $c_n = \dfrac{c_{n-1}}{2n^2 + 3n}$

$$y_2(x) = x^{3/2}\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \frac{x^4}{83160} + \cdots\right) = x^{3/2}\left[1 + 3\sum_{n=1}^{\infty}\frac{x^n}{n!(2n+3)!!}\right]$$

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20. With exponent $r_1 = 0$: $c_n = -\dfrac{2c_{n-1}}{3n^2 - n}$

$$y_1(x) = x^0\left(1 - x + \frac{x^2}{5} - \frac{x^3}{60} + \frac{x^4}{1320} - \cdots\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{n!\cdot 2\cdot 5\cdot\cdots\cdot(3n-1)}$$

With exponent $r_2 = \dfrac{1}{3}$: $c_n = -\dfrac{2c_{n-1}}{3n^2 + n}$

$$y_2(x) = x^{1/3}\left(1 - \frac{x}{2} + \frac{x^2}{14} - \frac{x^3}{210} + \frac{x^4}{5460} - \cdots\right) = x^{1/3}\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!\cdot 1\cdot 4\cdot\cdots\cdot(3n+1)}$$

The differential equations in Problems 21–24 are all of the form

$$Ax^2y'' + Bxy' + (C + Dx^2)y = 0 \tag{1}$$

with indical equation

$$\phi(r) = Ar^2 + (B - A)r + C = 0. \tag{2}$$

Substitution of $y = \Sigma c_n x^{n+r}$ into the differential equation yields

$$\phi(r)c_0x^r + \phi(r+1)c_1x^{r+1} + \sum_{n=2}^{\infty}\left[\phi(r+n)c_n + D\,c_{n-2}\right]x^{n+r} = 0. \tag{3}$$

In each of Problems 21–24 the exponents r_1 and r_2 do *not* differ by an integer. Hence when we substitute either $r = r_1$ or $r = r_2$ into Equation (*) above, we find that c_0 is arbitrary because $\phi(r)$ is then zero, that $c_1 = 0$ — because its coefficient $\phi(r+1)$ is then nonzero — and that

$$c_n = -\frac{Dc_{n-2}}{\phi(r+n)} = -\frac{Dc_{n-2}}{A(n+r)^2 + (B-A)(n+r) + C} \tag{4}$$

for $n \geq 2$. Thus this recurrence formula yields two linearly independent Frobenius series solutions when we apply it separately with $r = r_1$ and with $r = r_2$.

21. With exponent $r_1 = 1$: $c_1 = 0$, $c_n = \dfrac{2c_{n-2}}{n(2n+3)}$

$$y_1(x) = x^1\left(1 + \frac{x^2}{7} + \frac{x^4}{154} + \frac{x^6}{6930} + \cdots\right) = x\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\cdot 7\cdot 11\cdot\cdots\cdot(4n+3)}\right]$$

With exponent $r_2 = -\dfrac{1}{2}$: $c_1 = 0$, $c_n = \dfrac{2c_{n-2}}{n(2n-3)}$

$$y_2(x) = x^{-1/2}\left(1 + x^2 + \frac{x^4}{10} + \frac{x^6}{270} + \cdots\right) = \frac{1}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\cdot 1\cdot 5\cdot\cdots\cdot(4n-3)}\right]$$

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22. With exponent $r_1 = \frac{3}{2}$: $c_1 = 0,\quad c_n = -\frac{2c_{n-2}}{n(2n+5)}$

$$y_1(x) = x^{3/2}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} - \frac{x^6}{11934} + \cdots\right) = x^{3/2}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\cdot 9\cdot 13\cdot\cdots\cdot(4n+5)}\right]$$

With exponent $r_2 = -1$: $c_1 = 0,\quad c_n = -\frac{2c_{n-2}}{n(2n-5)}$

$$y_2(x) = x^{-1}\left(1 + x^2 - \frac{x^4}{6} + \frac{x^6}{126} - \frac{x^8}{5544} + \cdots\right) = \frac{1}{x}\left[1 + x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}x^{2n}}{n!\cdot 3\cdot 7\cdot\cdots\cdot(4n-5)}\right]$$

23. With exponent $r_1 = \frac{1}{2}$: $c_1 = 0,\quad c_n = \frac{c_{n-2}}{n(6n+7)}$

$$y_1(x) = x^{1/2}\left(1 + \frac{x^2}{38} + \frac{x^4}{4712} + \frac{x^6}{1215696} + \cdots\right) = \sqrt{x}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n!\cdot 19\cdot 31\cdot\cdots\cdot(12n+7)}\right]$$

With exponent $r_2 = -\frac{2}{3}$: $c_1 = 0,\quad c_n = \frac{c_{n-2}}{n(6n-7)}$

$$y_2(x) = x^{-2/3}\left(1 + \frac{x^2}{10} + \frac{x^4}{680} + \frac{x^6}{118320} + \cdots\right) = x^{-2/3}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n!\cdot 5\cdot 17\cdot\cdots\cdot(12n-7)}\right]$$

24. With exponent $r_1 = \frac{1}{3}$: $c_1 = 0,\quad c_n = -\frac{c_{n-2}}{n(3n+1)}$

$$y_1(x) = x^{1/3}\left(1 - \frac{x^2}{14} + \frac{x^4}{728} - \frac{x^6}{82992} + \cdots\right) = \sqrt[3]{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\cdot 7\cdot 13\cdot\cdots\cdot(6n+1)}\right]$$

With exponent $r_2 = 0$: $c_1 = 0,\quad c_n = -\frac{c_{n-2}}{n(3n-1)}$

$$y_2(x) = x^0\left(1 - \frac{x^2}{10} + \frac{x^4}{440} - \frac{x^6}{44880} + \cdots\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\cdot 5\cdot 11\cdot\cdots\cdot(6n-1)}$$

25. With exponent $r_1 = \frac{1}{2}$: $c_n = -\frac{c_{n-1}}{2n}$

$$y_1(x) = x^{1/2}\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots\right) = \sqrt{x}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!2^n} = \sqrt{x}\, e^{-x/2}$$

With exponent $r_2 = 0$: $c_n = -\frac{c_{n-1}}{2n-1}$

$$y_2(x) = x^0\left(1 - x + \frac{x^2}{3} - \frac{x^3}{15} + \frac{x^4}{105} - \cdots\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n-1)!!}$$

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26. With exponent $r_1 = \dfrac{1}{2}$: $c_1 = 0,\quad c_n = \dfrac{c_{n-2}}{n}$

$$y_1(x) = x^{1/2}\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{48}+\frac{x^8}{384}+\cdots\right) = \sqrt{x}\sum_{n=0}^{\infty}\frac{x^{2n}}{n!2^n} = \sqrt{x}\,e^{x^2/2}$$

With exponent $r_2 = 0$: $c_1 = 0,\quad c_n = \dfrac{2c_{n-2}}{2n-1}$

$$y_2(x) = x^0\left(1+\frac{2x^2}{3}+\frac{4x^4}{21}+\frac{8x^6}{231}+\frac{16x^8}{3465}+\cdots\right) = 1+\sum_{n=1}^{\infty}\frac{2^n x^{2n}}{3\cdot 7\cdot\cdots\cdot(4n-1)}$$

The differential equations in Problems 27–29 (after multiplication by x) and the one in Problem 31 are of the same form (1) above as those in Problems 21–24. However, now the exponents r_1 and $r_2 = r_1 - 1$ *do* differ by an integer. Hence when we substitute the smaller exponent $r = r_2$ into Equation (3), we find that c_0 and c_1 are *both* arbitrary, and that c_n is given (for $n \geq 2$) by the recurrence relation in (4). Thus the *smaller* exponent r_2 yields the general solution $y(x) = c_0 y_1(x) + c_1 y_2(x)$ in terms of the two linearly independent Frobenius series solutions $y_1(x)$ and $y_2(x)$.

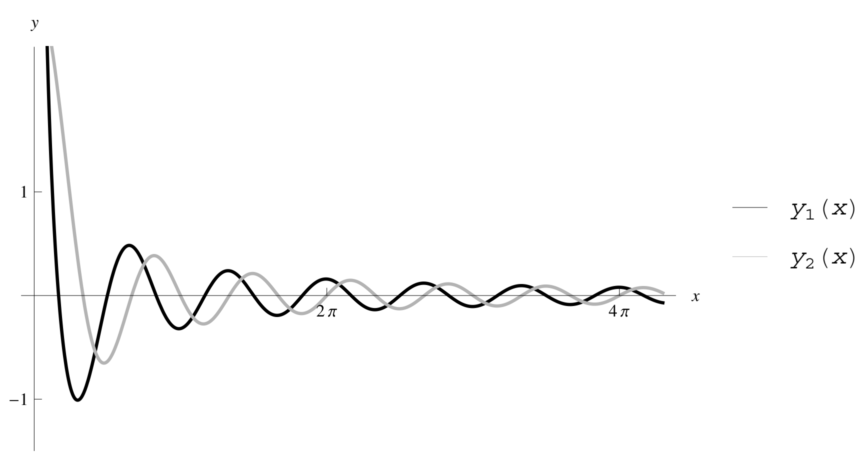
27. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $c_n = -\dfrac{9c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1-\frac{9x^2}{2}+\frac{27x^4}{8}-\frac{81x^6}{80}+\cdots\right)+\frac{c_1}{x}\left(x-\frac{3x^3}{2}+\frac{27x^5}{40}-\frac{81x^7}{560}+\cdots\right)$$

$$= \frac{c_0}{x}\left(1-\frac{9x^2}{2}+\frac{81x^4}{24}-\frac{729x^6}{720}+\cdots\right)+\frac{c_1}{3x}\left(3x-\frac{27x^3}{6}+\frac{243x^5}{120}-\frac{2187x^7}{5040}+\cdots\right)$$

$$y(x) = c_0\frac{\cos 3x}{x}+\frac{1}{3}c_1\frac{\sin 3x}{x}$$

The figure below shows the graphs of the independent solutions $y_1(x) = \dfrac{\cos 3x}{x}$ and $y_2(x) = \dfrac{\sin 3x}{x}$.



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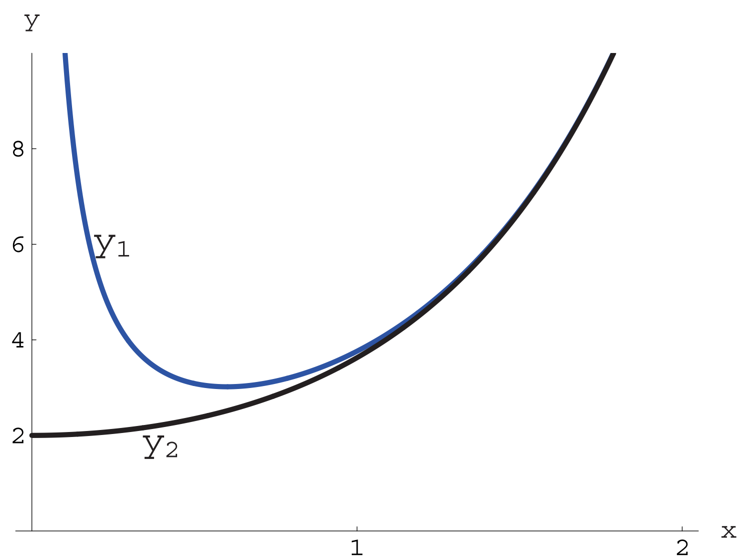
28. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $c_n = \dfrac{4c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1 + 2x^2 + \frac{2x^4}{3} + \frac{4x^6}{45} + \cdots\right) + \frac{c_1}{x}\left(x + \frac{2x^3}{3} + \frac{2x^5}{15} + \frac{4x^7}{315} + \cdots\right)$$

$$= \frac{c_0}{x}\left(1 + \frac{4x^2}{2} + \frac{16x^4}{24} + \frac{96x^6}{720} + \cdots\right) + \frac{c_1}{2x}\left(2x + \frac{8x^3}{6} + \frac{32x^5}{120} + \frac{128x^7}{5040} + \cdots\right)$$

$$y(x) = c_0\frac{\cosh 2x}{x} + \frac{1}{2}c_1\frac{\sinh 2x}{x}$$

The figure below shows the graphs of the two independent solutions $y_1(x) = \dfrac{\cosh 2x}{x}$ and $y_2(x) = \dfrac{\sinh 2x}{x}$.



29. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $c_n = -\dfrac{c_{n-2}}{4n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1 - \frac{x^2}{8} + \frac{x^4}{384} - \frac{x^6}{46080} + \cdots\right) + \frac{c_1}{x}\left(x - \frac{x^3}{24} + \frac{x^5}{1920} - \frac{x^7}{322560} + \cdots\right)$$

$$= \frac{c_0}{x}\left(1 - \frac{x^2}{2^2\cdot 2} + \frac{x^4}{2^4\cdot 24} - \frac{x^6}{2^6\cdot 720} + \cdots\right) + \frac{2c_1}{x}\left(\frac{x}{2} - \frac{x^3}{2^3\cdot 6} + \frac{x^5}{2^5\cdot 120} - \frac{x^7}{2^7\cdot 5040} + \cdots\right)$$

$$y(x) = \frac{c_0}{x}\cos\frac{x}{2} + \frac{2c_1}{x}\sin\frac{x}{2}$$

The figure at the top of the next page shows the graphs of the independent solutions $y_1(x) = \dfrac{\cos x/2}{x}$ and $y_2(x) = \dfrac{\sin x/2}{x}$.

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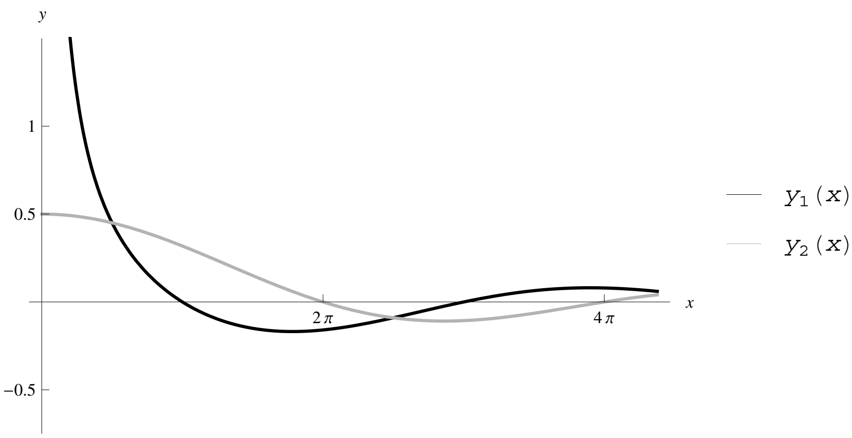


30. The given differential equation $xy'' - y' + 4x^3 y = 0$ has indicial equation $r^2 - 2r = r(r-2) = 0$, so its exponents are $r_1 = 2$ and $r_2 = 0$. Taking $r = 0$, substitution of the power series $y = \sum_{n=0}^{\infty} c_n x^n$ gives

$$-c_1 + 2c_3x^2 + (4c_0 + 8c_4)x^3 + (4c_1 + 15c_5)x^4 + (4c_2 + 24c_6)x^5 + (4c_3 + 35c_7)x^6 + (4c_4 + 48c_8)x^7 + (4c_5 + 63c_9)x^8 + \cdots = 0.$$

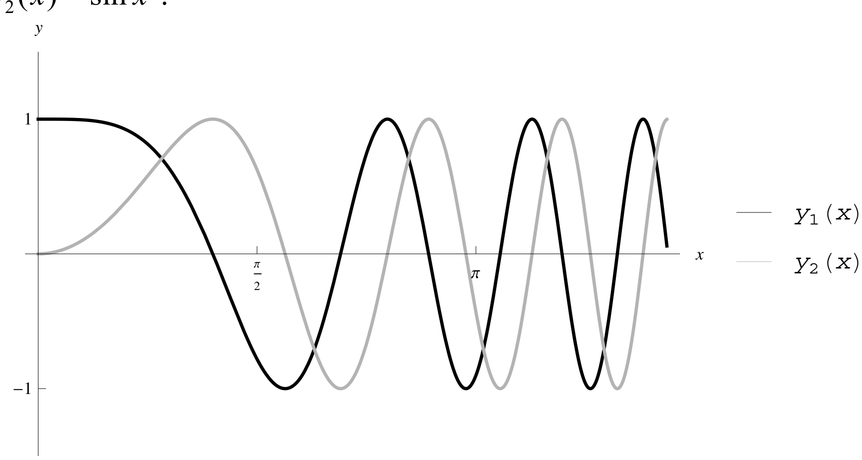
We see that $c_1 = c_3 = 0$ and

$$c_n = -\frac{4c_{n-4}}{n(n-2)} \quad \text{for } n \geq 4.$$

Hence the odd subscripts all vanish, and we obtain

$$y(x) = c_0 x\left(1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720} + \cdots\right) + c_2\left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} + \cdots\right)$$
$$y(x) = c_0 \cos x^2 + c_2 \sin x^2.$$

The figure below shows the graphs of the independent solutions $y_1(x) = \cos x^2$ and $y_2(x) = \sin x^2$.



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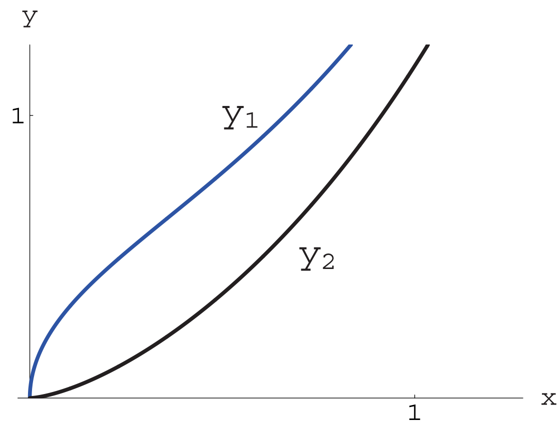
31. The given differential equation $4x^2y'' - 4xy' + (3-4x^2)y = 0$ has indicial equation $4r^2 - 8r + 3 \;=\; (2r-3)(2r-1) \;=\; 0$, so its exponents are $r_1 = 3/2$ and $r_2 = 1/2$. With $r = 3/2$, the recurrence relation $c_n \;=\; c_{n-2}/n(n-1)$ yields the general solution

$$y(x) \;=\; c_0 x^{1/2}\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\right) + c_1 x^{1/2}\left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right)$$

$$y(x) \;=\; c_0\sqrt{x}\cosh x + c_1\sqrt{x}\sinh x.$$

The figure below shows the graphs of the independent solutions $y_1(x) = \sqrt{x}\cosh x$ and $y_2(x) = \sqrt{x}\sinh x$.



32. The two indicial exponents are $r_1 \;=\; 1$ and $r_2 \;=\; -1/2$.

With $r_1 \;=\; 1$: Substitution of $y = x\sum c_n x^n$ in the differential equation yields

$$(5c_1 - c_0)x^2 + 14c_2x^3 + (c_2 + 27c_3)x^4 + (2c_3 + 44c_4)x^4 + (3c_4 + 65c_6)x^6 + \cdots \;=\; 0.$$

Hence we see that $c_1 \;=\; c_0/5$ and $c_2 = c_3 = c_4 = c_5 = \cdots 0$. Thus the series terminates and we obtain the polynomial solution

$$y_1(x) \;=\; x\left(1 + \frac{x}{5}\right) \;=\; x + \frac{x^2}{5}.$$

With $r_2 \;=\; -1/2$: We substitute $y = x^{-1/2}\sum c_n x^n$ and obtain the Frobenius solution

$$y_2(x) \;=\; \frac{1}{\sqrt{x}}\left(1 - \frac{5x}{2} - \frac{15x^2}{8} - \frac{5x^3}{48} + \frac{x^4}{384} + \cdots\right).$$

Remark: With appropriate interpretation of its result, the Mathematica **`DSolve`** function yields the two closed form solutions $y_1(x)$ and

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$$y_3(x) = x^{-1/2}e^{-x/2}\left(x^2+4x-2\right)+\sqrt{\frac{\pi}{2}}\,x\left(x+1\right)\operatorname{erf}\sqrt{\frac{x}{2}}.$$

Inquiring minds naturally want to know! The Mathematica **Series** command reveals the answer that $y_2(x) = -\frac{1}{2}y_3(x)$.

33. Exponents $r_1 = 1/2$ and $r_2 = -1$. With each exponent we find that c_0 is arbitrary and we can solve recursively for c_n in terms of c_{n-1}.

$$y_1(x) = \sqrt{x}\left(1+\frac{11x}{20}-\frac{11x^2}{224}+\frac{671x^3}{24192}-\frac{9577x^4}{387072}+\cdots\right)$$

$$y_2(x) = \frac{1}{x}\left(1+10x+5x^2+\frac{10x^3}{9}-\frac{7x^4}{18}+\cdots\right)$$

34. Exponents $r_1 = 1$ and $r_2 = -1/2$. With each exponent we find that $c_1 = 0$ and we can solve recursively for c_n in terms of c_{n-2}.

$$y_1(x) = x\left(1-\frac{x^2}{42}+\frac{x^4}{1320}-\frac{37x^6}{2494800}-\cdots\right)$$

$$y_2(x) = \frac{1}{\sqrt{x}}\left(1-\frac{7x^2}{24}+\frac{19x^4}{3200}-\frac{7661x^6}{43545600}+\cdots\right)$$

35. Substitution of $y = x^r\sum c_n x^n$ into the differential equation yields a result of the form

$$-rc_0x^{r-1}+(\cdots)x^r+(\cdots)x^{r+1}+\cdots = 0,$$

so we see immediately that $c_0 \neq 0$ implies that $r = 0$. Then substitution of the power series $y = \sum c_n x^n$ yields

$$(c_0-c_1)+(4c_1-2c_2)x+(9c_2-3c_3)x^2+(16c_3-4c_4)x^4+\cdots = 0$$

Evidently $c_n = nc_{n-1}$, so if $c_0 = 1$ it follows that $c_n = n!$ for $n \geq 1$. But the series $\sum n!x^n$ has zero radius of convergence, and hence converges only if $x = 0$. We therefore conclude that the given differential equation has *no* nontrivial Frobenius series solution.

36. **(a)** Substitution of $y = x^r\sum c_n x^n$ into the differential equation $x^2y'' + Ay' + By = 0$ yields a result of the form

$$Arc_0x^{r-1}+(\cdots)x^r+(\cdots)x^{r+1}+\cdots = 0,$$

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so we see immediately that $A \neq 0$ and $c_0 \neq 0$ imply that $r = 0$.

(b) Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' + Axy' + By = 0$ yields a result of the form

$$(Ar + B)c_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

so we see immediately that $c_0 \neq 0$ implies that $r = -B/A$.

(c) Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' + Ax^2 y' + By = 0$ yields a result of the form

$$Bc_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

which is impossible because both $c_0 \neq 0$ and $B \neq 0$. It follows that *no* Frobenius series can satisfy this equation.

37. Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' - xy' + y = 0$ yields a result of the form

$$(r-1)^2 c_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

so it follows that $r = 1$. But then substitution of $y = x\sum c_n x^n$ into the differential equation yields

$$c_1 x^2 + 4c_2 x^3 + 9c_3 x^4 + 16c_4 x^5 + 25c_5 x^6 + \cdots = 0,$$

so it follows that $c_1 = c_2 = c_3 = c_4 = \cdots = 0$. Hence $y(x) = c_0 x$.

38. Exponents $r_1 = 1/2$ and $r_2 = -1/2$; with $r = -1/2$: $c_n = -\dfrac{c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{\sqrt{x}}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) + \frac{c_1}{\sqrt{x}}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)$$

$$= \frac{c_0}{\sqrt{x}}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \frac{c_1}{\sqrt{x}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$y(x) = c_0 \frac{\cos x}{\sqrt{x}} + c_1 \frac{\sin 3x}{\sqrt{x}}$$

39. Exponents $r_1 = 1$ and $r_2 = -1$; with $r = +1$: $c_1 = 0$, $c_n = -\dfrac{c_{n-2}}{n(n+2)}$

$$y(x) = c_0 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + \frac{x^8}{737280} - \cdots\right)$$

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$$= c_0 x\left(1 - \frac{x^2}{2^2 1!2!} + \frac{x^4}{2^4 2!3!} - \frac{x^6}{2^6 3!4!} + \frac{x^8}{2^8 4!5!} - \cdots\right)$$

If $c_0 = 1/2$, then

$$y(x) = J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)}\left(\frac{x}{2}\right)^{2n}.$$

Now, consider the smaller exponent $r_2 = -1$. A Frobenius series with $r = -1$ is of the form $y = x^{-1}\sum_{n=0}^{\infty} c_n x^n$ with $c_0 \neq 0$. However, substitution of this series into Bessel's equation of order 1 gives

$$-c_1 + c_0 x + (c_1 + 3c_3)x^2 + (c_2 + 8c_4)x^3 + (c_3 + 15c_5)x^5 + \cdots = 0,$$

so it follows that $c_0 = 0$, after all. Thus Bessel's equation of order 1 does not have a Frobenius series solution with leading term $c_0 x^{-1}$. However, there is a little more here that meets the eye. We see further that c_2 is arbitrary and that $c_1 = 0$ and $c_n = c_{n-2}/n(n-2)$ for $n > 2$. It follows that our assumed Frobenius series $y = x^{-1}\sum_{n=0}^{\infty} c_n x^n$ actually reduces to

$$y(x) = c_2 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + \frac{x^8}{737280} - \cdots\right).$$

But this is the same as our series solution obtained above using the larger exponent $r = +1$ (calling the arbitrary constant c_2 rather than c_0).

SECTION 8.4

METHOD OF FROBENIUS—THE EXCEPTIONAL CASES

Each of the differential equations in Problems 1–6 is of (or can be written in) the form

$$xy'' + (A + Bx)y' + Cy = 0.$$

The origin is a regular singular point with exponents $r = 0$ and $r = 1 - A$, so if A is an integer then we have an exceptional case of the method of Frobenius. When we substitute $y = \Sigma c_n x^{n+r}$ in the differential equation we find that the coefficient of x^{n+r} is

$$[(n + r)^2 + (A - 1)(n + r)]c_n + [B(n + r) + C - B]c_{n-1} = 0. \qquad (*)$$

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Case 1: In each of Problems 1–4 we have $A \geq 2$ and $B = C$, so the larger exponent $r_1 = 0$ and the smaller exponent $r_2 = 1 - A = -N$ differ by a positive integer. When we substitute the smaller exponent $r = -N$ in Equation (*) above, it simplifies to

$$n(n - N)c_n + B(n - N)c_{n-1} = 0. \tag{1}$$

This equation determines $c_1, c_2, \cdots, c_{N-1}$ in terms of c_0, thereby yielding the solution

$$y_1(x) = x^{-N}(c_0 + c_1 x + \cdots + c_{N-1}x^{n-1}), \tag{2}$$

provided it is possible to choose $c_N = 0$. But when $n = N$, Equation (1) reduces to

$$0 \cdot c_N + 0 \cdot c_{N-1} = 0,$$

so c_N may be chosen arbitrarily. With $C_N = 0$ we get the terminating Frobenius series solution in (2). For $n > N$, Equation (1) yields the recurrence formula $c_n = -Bc_{n-1}/n$, which if $C_N \neq 0$ gives a second (non-terminating) Frobenius series solution of the form

$$y_2(x) = c_N + c_{N+1}x + c_{N+2}x^2 + \cdots. \tag{3}$$

Case 2: If $A \leq 0$ then the larger exponent $r_1 = 1 - A = N$ and the smaller exponent $r_2 = 0$ again differ by a positive integer. In Problems 5 and 6 we have this case with $B = -1$. When we substitute the smaller exponent $r = 0$ in Equation (*), it simplifies to

$$n(n - N)c_n - (n - C - 1)c_{n-1} = 0. \tag{4}$$

This equation determines $c_1, c_2, \cdots, c_{N-1}$ in terms of c_0. When $n = N$ it reduces to

$$0 \cdot c_N - (N - C - 1)c_{N-1} = 0. \tag{5}$$

If either $N - C - 1 = 0$ or $c_{N-1} = 0$ (the latter happens in Problem 5) then c_N can be chosen arbitrarily, and finally $c_{N+1}, c_{N+2}, \cdots$ are determined in terms of c_N. Thus we get *two* Frobenius series solutions

$$y_1 = c_0 + c_1 x + \cdots + c_{N-1}x^{N-1}, \qquad \text{(terminating)}$$

$$y_2 = c_N x^N + c_{N+1}x^{N+1} + \cdots. \qquad \text{(not terminating)}$$

On the other hand, if (as in Problem 6) neither $N - C - 1 = 0$ nor $c_{N-1} = 0$, then c_N cannot be chosen so as to satisfy Equation (5), and hence there is no Frobenius series solution corresponding to the smaller exponent $r_2 = 0$. We therefore find the *single* Frobenius series solution by substituting the larger exponent $r_1 = N$ in Equation (*) and using the resulting recurrence relation to determine $c_1, c_2, c_3, \cdots$ in terms of c_0.

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Problems 1–4 correspond to case 1 above. We give first the indicial roots and the critical index N, then the recurrence relation that defines c_n in terms of c_{n-1}, for both the N-term solution $y_1(x)$ in (2) and the non-terminating series solution $y_2(x)$ in (3).

1. $r_1 = 0,\ r_2 = -2,\ N = 2,\ c_n = \dfrac{c_{n-1}}{n};\qquad y_1(x) = x^{-2}(1+x);$

$$y_2(x) = 1 + \frac{x}{3} + \frac{x^2}{3\cdot 4} + \frac{x^3}{3\cdot 4\cdot 5} + \cdots = 1 + 2\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$$

2. $r_1 = 0,\ r_2 = -4,\ N = 4,\ c_n = \dfrac{c_{n-1}}{n};\qquad y_1(x) = x^{-4}\left(1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3\right)$

$$y_2(x) = 1 + \frac{x}{5} + \frac{x^2}{5\cdot 6} + \frac{x^3}{5\cdot 6\cdot 7} + \cdots = 1 + 24\sum_{n=1}^{\infty} \frac{x^n}{(n+4)!}$$

3. $r_1 = 0,\ r_2 = -4,\ N = 4,\ c_n = -\dfrac{3c_{n-1}}{n};\qquad y_1(x) = x^{-4}\left(1 - 3x + \dfrac{9}{2}x^2 - \dfrac{9}{2}x^3\right)$

$$y_2(x) = 1 - \frac{3x}{5} + \frac{3^2x^2}{5\cdot 6} - \frac{3^3x^3}{5\cdot 6\cdot 7} + \cdots = 1 + 24\sum_{n=1}^{\infty} \frac{(-1)^n 3^n x^n}{(n+4)!}$$

4. $r_1 = 0,\ r_2 = -5,\ N = 5,\ c_n = -\dfrac{3c_{n-1}}{5n}$

$$y_1(x) = x^{-5}\left(1 - \frac{3}{5}x + \frac{9}{50}x^2 - \frac{9}{250}x^3 + \frac{27}{5000}x^4\right)$$

$$y_2(x) = 1 - \frac{3x}{5\cdot 6} + \frac{3^2x^2}{5^2\cdot 6\cdot 7} - \frac{3^3x^3}{5^3\cdot 6\cdot 7\cdot 8} + \cdots = 1 + 120\sum_{n=1}^{\infty} \frac{(-1)^n 3^n x^n}{(n+5)!\ 5^n}$$

Problems 5 and 6 correspond to case 2 described above.

5. $r_1 = 5,\ r_2 = 0,\ N = 5,\ c_n = \dfrac{(n-4)c_{n-1}}{n(n-5)}$ for $n \neq 5$

$$y_1(x) = 1 + \frac{3}{4}x + \frac{1}{4}x^2 + \frac{1}{24}x^3$$

With $n = 5$ the recurrence relation is $0\cdot c_5 - c_4 = 0$. Because $c_4 = 0$ we can choose $c_5 = 1$ arbitrarily and proceed:

$$y_2(x) = x^5 + \frac{2x^6}{6} + \frac{3x^7}{6\cdot 7} + \frac{4x^8}{6\cdot 7\cdot 8} + \cdots = x^5\left[1 + 120\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+5)!}\right]$$

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6. Here $A = -3$, $B = -1$, $C = 1/2$, $r_1 = N = 4$, and $r_2 = 0$, so Equation (4) above is

$$n(n-4)c_n - (n - \tfrac{3}{2})c_{n-1} = 0.$$

Starting with $c_0 = 1$, this equation gives $c_1 = 1/6$, $c_2 = -1/48$, $c_3 = 1/96$. With $n = 4$ it reduces to

$$0 \cdot c_4 - \frac{7}{2} \cdot \frac{1}{96} = 0,$$

so c_4 cannot be chosen. We therefore start over by substituting $r_1 = 4$ in Equation (*) above and get the recurrence relation

$$c_n = \frac{2n+5}{2n(n+4)} c_{n-1}$$

for the coefficients in $y = x^4 \sum_{n=0}^{\infty} c_n x^n$. This yields the single Frobenius series solution

$$y_1(x) = x^4 \left(1 + \frac{7x}{2 \cdot 1 \cdot 5} + \frac{7 \cdot 9x^2}{2^2 \cdot 1 \cdot 2 \cdot 5.6} + \frac{7 \cdot 9 \cdot 11x^2}{2^3 \cdot 1 \cdot 2 \cdot 3 \cdot 5 \cdot 6 \cdot 7} + \cdots \right)$$

$$= x^4 \left(1 + \frac{8}{5} \sum_{n=1}^{\infty} \frac{(2n+5)!!\,x^n}{2^n n!(n+4)!} \right).$$

7. The indicial exponents are $r = -2,\ 1$. Substitution of $y = x^{-2} \sum_{n=0}^{\infty} c_n x^n$ in the differential equation leads to the recurrence relation

$$n(n-3)c_n + 3(n-3)c_{n-1} = 0$$

that reduces to $0 \cdot c_3 + 0 \cdot c_2 = 0$ when $n = 3$ so — having found c_1 and c_2 — c_3 can be chosen arbitrarily. With $c_0 = 2$ and $c_3 = 0$ we get the terminating Frobenius series

$$y_1(x) = x^{-2}(2 - 6x + 9x^2).$$

Starting afresh with $c_3 = 3/3! = 1/2$, the recurrence relation $c_n = -3c_{n-1}/n$ for $n > 3$ yields the second Frobenius series solution

$$y_2(x) = x^{-2} \left(\frac{3x^3}{3!} - \frac{3^2 x^4}{4!} + \frac{3^3 x^5}{5!} - \cdots \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n x^n}{(n+2)!}.$$

8. The exponents are $r = 0, 4$. When we substitute $y = \sum_{n=0}^{\infty} c_n x^n$ $\Sigma c_n x^n$ (corresponding to $r = 0$) in the differential equation we get the recurrence relation.

$$(n-4)c_n - (n-3)c_{n-1} = 0$$

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for $n \geq 1$. Starting with $c_0 = 3$, we compute $c_1 = 2$, $c_2 = 1$, and $c_3 = 0$. Because of the latter, we can select $c_4 = 0$ and get the terminating Frobenius series solution

$$y_1(x) = 3 + 2x + x^2.$$

But we also can choose $c_4 = 1$. Then our recurrence formula above yields $c_5 = 2$, $c_6 = 3$, $c_7 = 4, \cdots$. Hence the second Frobenius series solution is

$$y_2(x) = x^4(1 + 2x + 3x^2 + 4x^3 + \cdots) = x^4/(1-x)^2,$$

with the closed form coming from the derivative of the geometric series $1/(1-x) = \Sigma x^n$.

In Problems 11–15, we give first the Frobenius series solution $y_1(x)$ corresponding to the larger indicial exponent r_1 of the given differential equation. Then, writing the equation in the form $y'' + P(x)y' + Q(x)y = 0$, we apply the reduction of order formula

$$y_2(x) = \int \frac{\exp\left(-\int P(x)\,dx\right)}{y_1(x)^2}\,dx$$

to derive a second independent solution $y_2(x)$.

9. $r_1 = r_2 = 0$

$$y_1 = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \frac{x^8}{147456} + \cdots$$

$$P(x) = 1/x; \qquad \exp\left(-\int P(x)\,dx\right) = 1/x$$

$$y_2 = y_1 \int x^{-1} \cdot \left(1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \frac{x^8}{147456} + \cdots\right)^{-2} dx$$

$$= y_1 \int x^{-1}\left(1 + \frac{x^2}{2} + \frac{3x^4}{32} + \frac{5x^6}{576} + \frac{35x^8}{73728} + \cdots\right)^{-1} dx$$

$$= y_1 \int x^{-1}\left(1 - \frac{x^2}{2} + \frac{5x^4}{32} - \frac{23x^6}{576} + \frac{677x^8}{73728} + \cdots\right) dx$$

$$y_2 = y_1\left(\ln x - \frac{x^2}{4} + \frac{5x^4}{128} - \frac{23x^6}{3456} + \frac{677x^6}{589824} - \cdots\right)$$

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10. $r_1 = r_2 = 1$

$$y_1 = x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147456} - \cdots\right)$$

$$P(x) = -1/x; \qquad \exp\left(-\int P(x)\,dx\right) = x$$

$$y_2 = y_1 \int x^{-1} \cdot \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147456} - \cdots\right)^{-2} dx$$

$$= y_1 \int x^{-1}\left(1 - \frac{x^2}{2} + \frac{3x^4}{32} - \frac{5x^6}{576} + \frac{35x^8}{73728} - \cdots\right)^{-1} dx$$

$$= y_1 \int x^{-1}\left(1 + \frac{x^2}{2} + \frac{5x^4}{32} + \frac{23x^6}{576} + \frac{677x^8}{73728} + \cdots\right) dx$$

$$y_2 = y_1\left(\ln x + \frac{x^2}{4} + \frac{5x^4}{128} + \frac{23x^6}{3456} + \frac{677x^6}{589824} + \cdots\right)$$

11. $r_1 = r_2 = 2$

$$y_1 = x^2\left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \cdots\right)$$

$$P(x) = 1 - 3/x; \qquad \exp\left(-\int P(x)\,dx\right) = x^3 e^{-x}$$

$$y_2 = y_1 \int x^3 e^{-x} \cdot x^{-4}\left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \cdots\right)^{-2} dx$$

$$= y_1 \int x^{-1} e^{-x}\left(1 - 4x + 7x^2 - \frac{22x^3}{3} + \frac{16x^4}{3} - \cdots\right)^{-1} dx$$

$$= y_1 \int x^{-1}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right)\left(1 + 4x + 9x^2 + \frac{46x^3}{3} + \frac{67x^4}{3} + \cdots\right) dx$$

$$= y_1 \int x^{-1}\left(1 + 3x + \frac{11x^2}{2} + \frac{49x^3}{6} + \frac{87x^4}{8} + \cdots\right) dx$$

$$y_2 = y_1\left(\ln x + 3x + \frac{11x^2}{4} + \frac{49x^3}{18} + \frac{87x^4}{32} + \cdots\right)$$

12. $r_1 = 2, \quad r_2 = -1$

$$y_1 = x^2\left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \cdots\right)$$

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$$P(x) = 1; \qquad \exp\left(-\int P(x)\,dx\right) = e^{-x}$$

$$y_2 = y_1 \int e^{-x} \cdot x^{-4}\left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \cdots\right)^{-2} dx$$

$$= y_1 \int x^{-4}e^{-x}\left(1 - x + \frac{11x^2}{20} - \frac{13x^3}{60} + \frac{569x^4}{8400} - \cdots\right)^{-1} dx$$

$$= y_1 \int x^{-4}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right)\left(1 + x + \frac{9x^2}{20} + \frac{7x^3}{60} + \frac{19x^4}{1050} + \cdots\right) dx$$

$$= y_1 \int x^{-4}\left(1 - \frac{x^2}{20} + \frac{x^4}{700} - \frac{47x^6}{1512000} + \cdots\right) dx$$

$$y_2 = y_1\left(-\frac{1}{3x^3} + \frac{1}{20x} + \frac{x}{700} - \frac{47x^3}{4536000} + \cdots\right) \qquad \text{[no logarithmic term]}$$

13. $r_1 = 3, \quad r_2 = 1$

$$y_1 = x^3\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \cdots\right)$$

$$P(x) = 2 - 3/x; \qquad \exp\left(-\int P(x)\,dx\right) = x^3e^{-2x}$$

$$y_2 = y_1 \int x^3e^{-2x} \cdot x^{-6}\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \cdots\right)^{-2} dx$$

$$= y_1 \int x^{-3}e^{-2x}\left(1 - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \cdots\right)^{-1} dx$$

$$= y_1 \int x^{-3}\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \cdots\right)\left(1 + 4x + 8x^2 + \frac{32x^3}{3} + \frac{32x^4}{3} + \cdots\right) dx$$

$$= y_1 \int x^{-3}\left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots\right) dx$$

$$y_2 = y_1\left(2\ln x - \frac{1}{2x^2} - \frac{2}{x} + \frac{4x}{3} + \frac{x^2}{3} + \cdots\right)$$

14. $r_1 = 2, \quad r_2 = -2$

$$y_1 = x^2\left(1 - \frac{2x}{5} + \frac{x^2}{10} - \frac{2x^3}{105} + \frac{x^4}{336} - \frac{x^5}{2520} + \cdots\right)$$

$$P(x) = 1 + 1/x; \qquad \exp\left(-\int P(x)\,dx\right) = x^{-1}e^{-x}$$

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$$y_2 = y_1 \int x^{-1}e^{-x} \cdot x^{-4}\left(1-\frac{2x}{5}+\frac{x^2}{10}-\frac{2x^3}{105}+\frac{x^4}{336}-\frac{x^5}{2520}+\cdots\right)^{-2} dx$$

$$= y_1 \int x^{-5}e^{-x}\left(1-\frac{4x}{4}+\frac{9x^2}{25}-\frac{62x^3}{525}+\frac{131x^4}{4200}-\frac{11x^5}{1575}+\cdots\right)^{-1} dx$$

$$= y_1 \int x^{-5}\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+\cdots\right)\cdot \left(1+\frac{4x}{5}+\frac{7x^2}{25}+\frac{142x^3}{2625}+\frac{121x^4}{21000}+\frac{46x^5}{196875}-\cdots\right)dx$$

$$= y_1 \int x^{-5}\left(1-\frac{x}{5}-\frac{x^2}{50}+\frac{13x^3}{1750}-\frac{29x^5}{196875}+\cdots\right)dx$$

$$y_2 = y_1\left(-\frac{1}{4x^4}+\frac{15}{x^3}+\frac{1}{100x^2}-\frac{13}{1750x}+0\cdot\ln x-\frac{29x}{196875}+\cdots\right)$$

Thus the second solution $y_2(x)$ contains no logarithmic term.

15. $r_1 = r_2 = 0$

$$J_0(x) = 1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147456}-\cdots$$

$$P(x) = 1/x; \qquad \exp\left(-\int P(x)\,dx\right) = 1/x$$

$$y_2(x) = J_0(x)\int x^{-1}\cdot\left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147456}-\cdots\right)^{-2} dx$$

$$= J_0(x)\int x^{-1}\left(1-\frac{x^2}{2}+\frac{3x^4}{32}-\frac{5x^6}{576}+\frac{35x^8}{73728}-\cdots\right)^{-1} dx$$

$$= J_0(x)\int x^{-1}\left(1+\frac{x^2}{2}+\frac{5x^4}{32}+\frac{23x^6}{576}+\frac{677x^8}{73728}+\cdots\right)dx$$

$$= J_0(x)\left(\ln x+\frac{x^2}{4}+\frac{5x^4}{128}+\frac{23x^6}{3456}+\frac{677x^6}{589824}-\cdots\right)$$

$$= J_0(x)\ln x + \left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147456}-\cdots\right)\left(\frac{x^2}{4}+\frac{5x^4}{128}+\frac{23x^6}{3456}+\frac{677x^6}{589824}-\cdots\right)$$

$$y_2(x) = J_0(x)\ln x+\frac{x^2}{4}-\frac{3x^4}{128}+\frac{11x^6}{13284}-\cdots$$

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16. The indicial exponents are $r = \pm\frac{3}{2}$. We start with the larger exponent $r_1 = +\frac{3}{2}$ and substitute $y = x^{3/2}\sum_{n=0}^{\infty} a_n x^n$ into Bessel's equation of order $\frac{3}{2}$. We find that $c_1 = 0$, and then the recurrence relation

$$a_n = -\frac{a_{n-2}}{n(n+3)},$$

implies that all odd subscripts vanish. Starting with $a_0 = 1$, this recurrence relation yields in the usual manner the first solution

$$\begin{aligned} y_1(x) &= x^{3/2}\left[1 - \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 7} - \frac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 7\cdot 8} + \cdots\right] \\ &= x^{3/2}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!\cdot 5\cdot 7\cdot\cdots\cdot(2n+3)}\right]. \end{aligned}$$

Now we start afresh with the smaller exponent $r_1 = -\frac{3}{2}$ and substitute $y = x^{-3/2}\sum_{n=0}^{\infty} b_n x^n$ into Bessel's equation of order $\frac{3}{2}$. This time, we find that $n(n-3)b_n + b_{n-2} = 0$ for $n \ge 2$. We can satisfy the critical case $0\cdot b_3 + b_1 = 0$ by choosing $b_1 = 0$, which then implies that all odd coefficients vanish. Then the recurrence relation

$$b_n = -\frac{b_{n-2}}{n(n-3)}$$

yields routinely the second solution

$$\begin{aligned} y_2(x) &= x^{-3/2}\left[1 - \frac{x^2}{2\cdot(-1)} + \frac{x^4}{2\cdot 4\cdot(-1)\cdot 1} - \frac{x^6}{2\cdot 4\cdot 6\cdot(-1)\cdot 1\cdot 3} + \cdots\right] \\ &= x^{-3/2}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^n n!\cdot(-1)\cdot 1\cdot 3\cdot\cdots\cdot(2n-3)}\right]. \end{aligned}$$

17. The given first solution

$$y_1(x) = x e^x = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right)$$

can be derived by starting with the single exponent $r = 1$, substituting $y = x\sum_{n=0}^{\infty} c_n x^n$ into the differential equation, and calculating successive coefficients recursively as usual. We can verify the alleged second solution by applying the method of reduction of order as in Problems 9–14:

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$$P(x) = -1 - \frac{1}{x}; \qquad \exp\left(-\int P(x)\,dx\right) = x\,e^x$$

$$y_2 = y_1 \int x\,e^x \cdot \left(x\,e^x\right)^{-2} dx = y_1 \int x^{-1}\,e^{-x}\,dx$$

$$= y_1 \int x^{-1}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right) dx$$

$$= y_1 \left(\ln x - x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + \cdots\right)$$

$$= y_1 \ln x + \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right)\left(-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + \cdots\right)$$

$$= y_1 \ln x - \left(x^2 + \frac{3x^3}{4} + \frac{11x^4}{36} + \frac{25x^5}{288} + \frac{137x^6}{7200} + \cdots\right)$$

$$y_2(x) = x\,e^x \ln x - \sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n!}$$

18. When we substitute

$$y(x) = Cy_1 \ln x + \sum_{n=0}^{\infty} b_n x^n$$

in the differential equation $xy'' - x = 0$ we find that $b_1 = -b_0 = -C$ and that

$$n(n+1)b_{n+1} - b_n = -\frac{(2n+1)C}{n!(n+1)!}$$

for $n \geq 1$. To solve this recurrence relation we take $C = 1$ and substitute $b_n = c_n / (n-1)!n!$. The result is

$$c_{n+1} - c_n = -\frac{2n+1}{n(n+1)} = -\frac{1}{n} - \frac{1}{n+1}.$$

Starting with $c_1 = b_1 = -1$, it follows readily by induction on n that $c_n = -(H_n + H_{n+1})$.

SECTION 8.5

BESSEL'S EQUATION

Of course Bessel's equation is the most important special ordinary differential equation in mathematics, and every student should be exposed at least to Bessel functions of the first kind. Though Bessel functions of integral order can be treated without the gamma function, the

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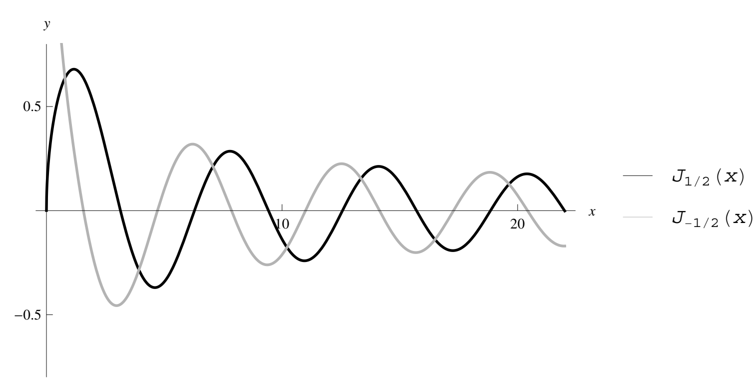
subsection on the gamma function is also needed for Chapter 7 on Laplace transforms. The final subsections on Bessel function identities and the parametric Bessel equation will not be needed until Section 10.4, and therefore may be considered optional at this point in the course.

1. $$J_0'(x) = D_x\left(1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right) = \sum_{m=1}^{\infty}\frac{(-1)^m 2m\, x^{2m-1}}{2^{2m}(m!)^2}$$
$$= \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-1}(m-1)!(m!)} = \sum_{m=0}^{\infty}\frac{(-1)^{m+1} x^{2m+1}}{2^{2m+1}(m)!(m+1!)}$$
$$= -\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}(m)!(m+1!)} = -J_1(x)$$

2. **(a)** $$\Gamma\left(\frac{2n+1}{2}\right) = \frac{2n-1}{2}\cdot\Gamma\left(\frac{2n-1}{2}\right)$$
$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\Gamma\left(\frac{2n-3}{2}\right)$$
$$\vdots$$
$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\cdot\left(\frac{1}{2}\right)$$
$$= \frac{(2n-1)(2n-3)\cdot\cdots\cdot 3\cdot 1}{2^n}\cdot\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

(b) $$J_{1/2}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+\frac{1}{2}}}{m!\,\Gamma(m+\frac{3}{2})2^{2m+\frac{1}{2}}} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{m!\,2^{-m-1}(2m+1)!!\,\sqrt{\pi}\,2^{2m+1}}$$
$$= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2\cdot 4\cdot\cdots\cdot 2m)(1\cdot 3\cdot\cdots\cdot(2m+1))}$$
$$= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}}\sin x$$
$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad \text{similarly}$$ (See the figure below for the graphs.)



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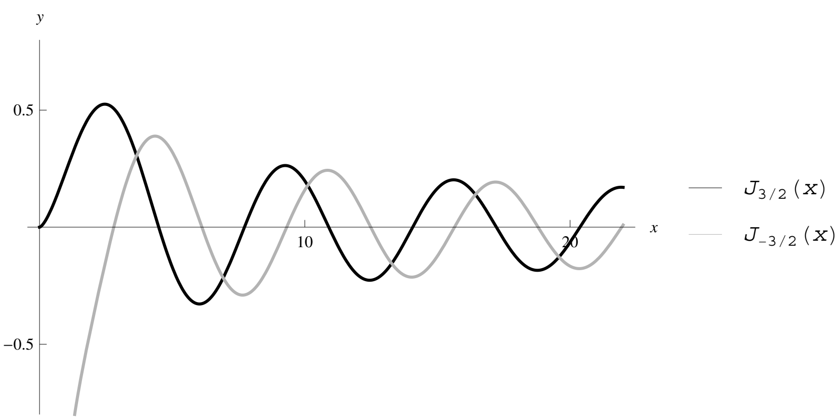
3. **(a)** $$\Gamma\left(m+\frac{2}{3}\right) = \Gamma\left(\frac{3m+2}{3}\right) = \frac{3m-1}{3}\cdot\frac{3m-4}{3}\cdot\Gamma\left(\frac{3m-4}{3}\right)$$
$$= \frac{3m-1}{3}\cdot\frac{3m-4}{3}\cdot\cdots\cdot\frac{5}{3}\cdot\frac{2}{3}\cdot\Gamma\left(\frac{2}{3}\right) = \frac{2\cdot5\cdot8\cdots\cdot(3m-1)}{3^m}\Gamma\left(\frac{2}{3}\right)$$

(b) $$J_{-1/3}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+2/3)}\left(\frac{x}{2}\right)^{2m-1/3} = \frac{(x/2)^{-1/3}}{\Gamma(2/3)}\sum_{m=0}^{\infty}\frac{(-1)^m 3^m x^{2m}}{m!\cdot2\cdot3\cdot8\cdots\cdot(3m-1)}$$

4. With $p = 1/2$ in Equation (26) in the text we have

$$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$
$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\left(\sin x - x\cos x\right) = \sqrt{\frac{2}{\pi x^3}}\left(\sin x - x\cos x\right)$$

The figure below shows the graphs of $J_{3/2}(x)$ and $J_{-3/2}(x)$.



5. Starting with $p = 3$ in Equation (26) we get

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) = \frac{6}{x}\left[\frac{4}{x}J_2(x) - J_1(x)\right] - J_2(x)$$
$$= \left(\frac{24}{x^2}-1\right)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \frac{6}{x}J_1(x)$$
$$= \frac{x^2-24}{x^2}J_0(x) + \frac{8(6-x^2)}{x^3}J_1(x)$$

8. When we carry out the differentiations indicated in Equations (22) and (23) in the text, we get

$$p\,x^{p-1}J_p(x) + x^p J_p'(x) = x^p J_{p-1}(x),$$
$$-p\,x^{-p-1}J_p(x) + x^{-p}J_p'(x) = -x^p J_{p+1}(x).$$

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When we solve these two equations for $J'_p(x)$ we get Equations (24) and (25) in the text.

10. When we add equations (24) and (25) we get

$$J'_p(x) = \frac{1}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right],$$

so

$$J''_p(x) = \frac{1}{2}\left[J'_{p-1}(x) - J'_{p+1}(x)\right].$$

Replacing p with $p-1$ and with $p+1$ in the first equation, we get

$$J'_{p-1}(x) = \frac{1}{2}\left[J_{p-2}(x) - J_p(x)\right]$$

and

$$J'_{p+1}(x) = \frac{1}{2}\left[J_p(x) - J_{p+2}(x)\right].$$

When we use these equations to substitute for $J'_{p-1}(x)$ and $J_{p+1}(x)$ in the equation for $J''_p(x)$ above, we find that

$$J''_p(x) = \frac{1}{4}\left[J_{p-2}(x) - 2J_p(x) + J_{p+2}(x)\right].$$

11. $\Gamma(p+m+1) = (p+m)(p+m-1)\cdot\dots\cdot(p+2)(p+1)\Gamma(p+1)$, so

$$\begin{aligned} J_p(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(p+m+1)}\left(\frac{x}{2}\right)^{2m+p} \\ &= \frac{(x/2)^p}{\Gamma(p+1)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(p+1)(p+2)\cdot\dots\cdot(p+m)}\left(\frac{x}{2}\right)^{2m}. \end{aligned}$$

12. Substitution of the power series of Problem 11 yields

$$y(x) = x^2 \cdot \frac{x^{5/2}(A+\cdots) + x^{-5/2}(B+\cdots)}{x^{1/2}(C+\cdots) + x^{-1/2}(D+\cdots)} = \frac{x^5(A+\cdots)+(B+\cdots)}{x(C+\cdots)+(D+\cdots)}$$

where $A = 1/(2^{5/2}\Gamma(7/2))$, $B = 1/(2^{-5/2}\Gamma(-3/2),)$ $C = 1/(2^{1/2}\Gamma(3/2))$, and $D = (1/2^{-1/2})\Gamma(1/2)$. Hence

$$y(0) = \frac{0\cdot(A+\cdots)+(B+\cdots)}{0\cdot(C+\cdots)+(D+\cdots)} = \frac{B}{D} = \frac{2^{-1/2}\Gamma(1/2)}{2^{-5/2}\Gamma(-3/2)} = \frac{2^2\Gamma(1/2)}{(4/3)\Gamma(1/2} = 3.$$

The graph of $y(x)$ shown at the top of the next page corroborates this value.

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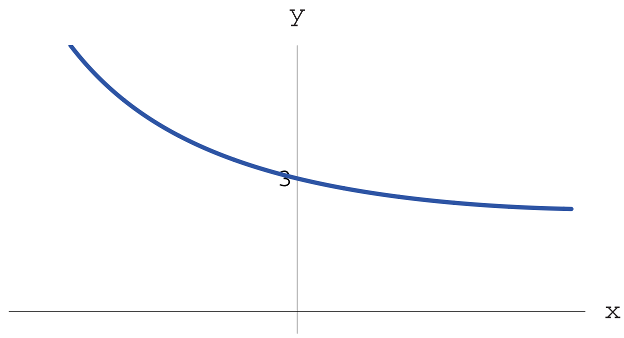

In Problems 13–21 we use a conspicuous dot • to indicate our choice of u and dv in the integration by parts formula $\int u \bullet dv = uv - \int v\,du$. We use repeatedly the facts (from Example 1) that $\int xJ_0(x)\,dx = xJ_1(x) + C$ and $\int J_1(x)\,dx = -J_0(x) + C$.

13.
$$\begin{aligned}\int x^2 J_0(x)\,dx &= \int x \bullet xJ_0(x)\,dx \\ &= x^2 J_1(x) - \int x \bullet J_1(x)\,dx \\ &= x^2 J_1(x) - \left(-xJ_0(x) + \int J_0(x)\,dx\right) \\ &= x^2 J_1(x) + xJ_0(x) - \int J_0(x)\,dx + C\end{aligned}$$

14.
$$\begin{aligned}\int x^3 J_0(x)\,dx &= \int x^2 \bullet xJ_0(x)\,dx \\ &= x^3 J_1(x) - 2\int x^2 \bullet J_1(x)\,dx \\ &= x^3 J_1(x) - 2\left(-x^2 J_0(x) + 2\int xJ_0(x)\,dx\right) \\ &= x^3 J_1(x) + 2x^2 J_0(x) - 4xJ_1(x) + C = (x^3 - 4x)J_1(x) + 2x^2 J_0(x) + C\end{aligned}$$

15.
$$\begin{aligned}\int x^4 J_0(x)\,dx &= \int x^3 \bullet xJ_0(x)\,dx \\ &= x^4 J_1(x) - 3\int x^3 \bullet J_1(x)\,dx \\ &= x^4 J_1(x) - 3\left(-x^3 J_0(x) + 3\int x \bullet xJ_0(x)\,dx\right) \\ &= x^4 J_1(x) + 3x^3 J_0(x) - 9\left(x^2 J_1(x) - \int x \bullet J_1(x)\,dx\right) \\ &= x^4 J_1(x) + 3x^3 J_0(x) - 9x^2 J_1(x) + 9\left(-xJ_0(x) + \int J_0(x)\,dx\right) \\ &= (x^4 - 9x^2)J_1(x) + (3x^3 - 9x)J_0(x) + 9\int J_0(x)\,dx + C\end{aligned}$$

16. $\int xJ_1(x)\,dx = \int x \bullet J_1(x)\,dx = -xJ_0(x) + \int J_0(x)\,dx + C$

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17. $$\int x^2 J_1(x)\,dx = \int x^2 \bullet J_1(x)\,dx$$
$$= -x^2 J_0(x) + 2\int xJ_0(x)\,dx = -x^2 J_0(x) + 2xJ_1(x) + C$$

18. $$\int x^3 J_1(x)\,dx = \int x^3 \bullet J_1(x)\,dx$$
$$= -x^3 J_0(x) + 3\int x \bullet xJ_0(x)\,dx$$
$$= -x^3 J_0(x) + 3\left(x^2 J_1(x) - \int x \bullet J_1(x)\,dx\right)$$
$$= -x^3 J_0(x) + 3x^2 J_1(x) - 3\left(-xJ_0(x) + \int J_0(x)\,dx\right)$$
$$= (-x^3 + 3x)J_0(x) + 3x^2 J_1(x) - 3\int J_0(x)\,dx + C$$

19. $$\int x^4 J_1(x)\,dx = \int x^4 \bullet J_1(x)\,dx$$
$$= -x^4 J_0(x) + 4\int x^2 \bullet xJ_0(x)\,dx$$
$$= -x^4 J_0(x) + 4\left(x^3 J_1(x) - 2\int x^2 \bullet J_1(x)\,dx\right)$$
$$= -x^4 J_0(x) + 4x^3 J_1(x) - 8\left(-x^2 J_0(x) + 2\int xJ_0(x)\,dx\right)$$
$$= (-x^4 + 8x^2)J_0(x) + (4x^3 - 16x)J_1(x) + C$$

20. With $p = 1$, Eq. (23) in the text gives $\int x^{-1} J_2(x)\,dx = -x^{-1} J_1(x) + C$. Hence

$$\int J_2(x)\,dx = \int x \bullet x^{-1} J_2(x)\,dx$$
$$= x\left(-x^{-1} J_1(x)\right) + \int x^{-1} J_1(x) = -J_1(x) + \int x^{-1} J_1(x)\,dx.$$

But Eq. (26) with $p = 1$ gives $x^{-1} J_1(x) = \tfrac{1}{2}\left[J_0(x) + J_2(x)\right]$, so

$$\int J_2(x)\,dx = -J_1(x) + \tfrac{1}{2}\int J_0(x)\,dx + \tfrac{1}{2}\int J_2(x)\,dx.$$

Finally, we can solve this last equation for

$$\int J_2(x)\,dx = -2J_1(x) + \int J_0(x)\,dx + C.$$

21. With $p = 2$, Eq. (23) in the text gives $\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x) + C$. Hence

$$\int J_3(x)\,dx = \int x^2 \bullet x^{-2} J_3(x)\,dx$$
$$= x^2\left(-x^{-2} J_2(x)\right) + 2\int x^{-1} J_2(x)\,dx$$
$$= -J_2(x) - \frac{2}{x} J_1(x) + C \qquad \text{(by Example 3)}$$

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$$= -\left(\frac{2}{x}J_1(x) - J_0(x)\right) - \frac{2}{x}J_1(x) + C \qquad \text{(By Eq. (26) with } p = 1)$$

$$= J_0(x) - \frac{4}{x}J_1(x) + C.$$

22. Let us define

$$g(x) = \int_0^\pi \cos(x\sin\theta)\,d\theta$$

and note first that

$$g(0) = \int_0^\pi \cos(0)\,d\theta = \pi = \pi J_0(0).$$

Differentiation under the integral sign yields

$$g'(x) = -\int_0^\pi \sin(x\sin\theta)\sin\theta\,d\theta.$$

When we integrate by parts with

$$u = \sin(x\sin\theta) \qquad dv = \sin\theta\,d\theta$$
$$du = (x\cos\theta)\cos(x\sin\theta)\,d\theta \qquad v = -\cos\theta$$

we get

$$g'(x) = -x\int_0^\pi \cos^2\theta\,\cos(x\sin\theta)\,d\theta.$$

But differentiation of the first equation for $g'(x)$ yields

$$g''(x) = -\int_0^\pi \sin^2\theta\cos(x\sin\theta)\,d\theta.$$

Finally, because $\cos^2\theta + \sin^2\theta = 1$, it follows that

$$g''(x) + \frac{1}{x}g'(x) = -\int_0^\pi \cos(x\sin\theta)\,d\theta = -g(x).$$

Thus $y = g(x)$ satisfies Bessel's equation of order zero in the form $y'' + (1/x)y' + y = 0$. Therefore the function g takes the form

$$g(x) = a\,J_0(x) + b\,Y_0(x).$$

Since $g(0) = \pi$ is finite and $J_0(0) = 1$, we must have $a = \pi$ and $b = 0$, so $g(x) = \pi J_0(x)$, as desired.

23. This is a special case of the discussion below in Problem 24.

24. Given an integer $n \geq 1$, let us define

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$$g_n(x) = \int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta.$$

Differentiation yields

$$g_n'(x) = \int_0^\pi \sin(n\theta - x\sin\theta)\sin\theta\,d\theta.$$

Integration by parts with $u = \sin(n\theta - x\sin\theta)$ and $dv = \sin\theta\,d\theta$ yields

$$g_n'(x) = n\int_0^\pi \cos\theta\,\cos(n\theta - x\sin\theta)\,d\theta - x\int_0^\pi \cos^2\theta\,\cos(n\theta - x\sin\theta)\,d\theta.$$

But differentiation of the first equation for $g_n'(x)$ yields

$$g_n''(x) = -\int_0^\pi \sin^2\theta\,\cos(n\theta - x\sin\theta)\,d\theta.$$

It follows that

$$\begin{aligned} g_n''(x) + \frac{1}{x}g_n'(x) &= -g_n(x) + \frac{n}{x}\int_0^\pi \cos\theta\,\cos(n\theta - x\sin\theta)\,d\theta \\ &= -g_n(x) - \frac{n}{x^2}\int_0^\pi \left[(n - x\cos\theta) - n\right]\cos(n\theta - x\sin\theta)\,d\theta \\ &= -g_n(x) - \frac{n}{x^2}\left[\sin(n\theta - x\sin\theta\right]_0^\pi + \frac{n^2}{x^2}g_n(x) = -\left(1 - \frac{n^2}{x^2}\right)g_n(x). \end{aligned}$$

Upon equating the first and last members of this continued inequality and multiplying by x^2, we see that $y = g_n(x)$ satisfies Bessel's equation of order $n \geq 1$. The initial values of $g_n(x)$ are

$$g_n(0) = \int_0^\pi \cos(n\theta)\,d\theta = 0 \quad\text{and}\quad g_n'(0) = \int_0^\pi \sin(\theta)\sin(n\theta)\,d\theta = 0.$$

If $n = 1$ then $g_1'(0) = \pi/2$, whereas $g_n'(0) = 0$ if $n \geq 1$. In either case the values of $g_n(0)$ and $g_n'(0)$ are π times those of $J_n(0)$ and $J_n'(0)$, respectively. Now we know from the general solution of Bessel's equation that $g_n(x) = c\,J_n(x)$ for some constant c. If $n = 1$ then the fact that

$$\pi/2 = g_n'(0) = c\,J_1'(0) = c/2$$

implies that $c = \pi$, as desired. But if $n > 1$ the fact that

$$0 = g_n'(0) = c\,J_n'(0) = c\cdot 0$$

does not suffice to determine c.

26. The graph shown at the top of the next page illustrates the interlaced zeros of the Bessel functions $J_{10}(x)$ and $J_{11}(x)$.

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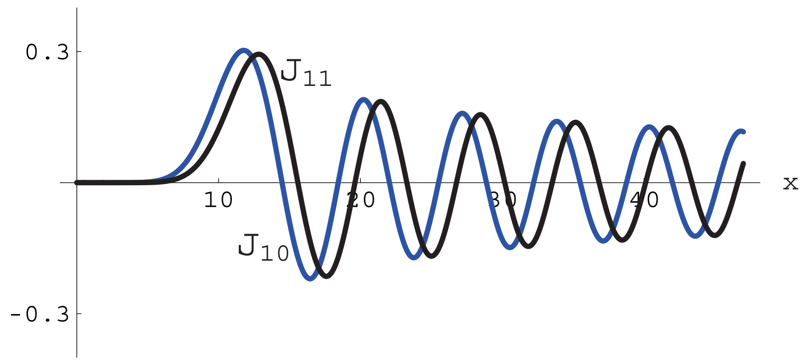

SECTION 8.6

APPLICATIONS OF BESSEL FUNCTIONS

Problems 1–12 are routine applications of the theorem in this section. In each case it is necessary only to identify the coefficients A, B, C and the exponent q in the differential equation

$$x^2 y'' + Axy' + (B + Cx^q)y = 0. \tag{1}$$

Then we can calculate the values

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{q}{2}, \quad k = \frac{2\sqrt{C}}{q}, \quad p = \frac{\sqrt{(1-A)^2 - 4B}}{q} \tag{2}$$

and finally write the general solution

$$y(x) = x^\alpha \left[c_1 J_p(kx^\beta) + c_2 J_{-p}(kx^\beta) \right] \tag{3}$$

specified in Theorem 1 of this section. This is a "template procedure" that we illustrate only in a couple of problems.

1. We have $A = -1, B = 1, C = 1, q = 2$ so

$$\alpha = \frac{1-(-1)}{2} = 1, \quad \beta = \frac{2}{2} = 1, \quad k = \frac{2\sqrt{1}}{2} = 1, \quad p = \frac{\sqrt{(1-(-1))^2 - 4(1)}}{2} = 0,$$

so our general solution is $y(x) = x[c_1 J_0(x) + c_2 Y_0(x)]$, using $Y_0(x)$ because $p = 0$ is an integer.

2. $y(x) = x^{-1}[c_1 J_1(x) + c_2 Y_1(x)]$

3. $y(x) = x[c_1 J_{1/2}(3x^2) + c_2 J_{-1/2}(3x^2)]$

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4. $y(x) = x^3[c_1J_2(2x^{1/2}) + c_2Y_2(2x^{1/2})]$

5. To match the given equation with Eq. (1) above, we first divide through by the leading coefficient 16 to obtain the equation

$$x^2y'' + \frac{5}{3}xy' + \left(-\frac{5}{36} + \frac{1}{4}x^3\right)y = 0$$

with $A = 5/3, B = -5/36, C = 1/4,$ and $q = 3.$ Then

$$\alpha = \frac{1-5/3}{3} = -\frac{1}{3}, \quad \beta = \frac{3}{2}, \quad k = \frac{2\sqrt{1/4}}{3} = \frac{1}{3}, \quad p = \frac{\sqrt{(1-5/3)^2 - 4(-5/36)}}{3} = \frac{1}{3},$$

so our general solution is $y(x) = x^{-1/3}[c_1J_{1/3}(x^{3/2}/3) + c_2J_{-1/3}(x^{3/2}/3)]$.

6. $y(x) = x^{-1/4}[c_1J_0(2x^{3/2}) + c_2Y_0(2x^{3/2})]$

7. $y(x) = x^{-1}[c_1J_0(x) + c_2Y_0(x)]$

8. $y(x) = x^2[c_1J_1(4x^{1/2}) + c_2Y_1(4x^{1/2})]$

9. $y(x) = x^{1/2}[c_1J_{1/2}(2x^{3/2}) + c_2J_{-1/2}(2x^{3/2})]$

10. $y(x) = x^{-1/4}[c_1J_{3/2}(2x^{5/2}/5) + c_2J_{-3/2}(2x^{5/2}/5)]$

11. $y(x) = x^{1/2}[c_1J_{1/6}(x^3/3) + c_2J_{-1/6}(x^3/3)]$

12. $y(x) = x^{1/2}[c_1J_{1/5}(4x^{5/2}/5) + c_2J_{-1/5}(4x^{5/2}/5)]$

13. We want to solve the equation $xy'' + 2y' + xy = 0$. If we rewrite it as

$$x^2y'' + 2xy' + x^2y = 0$$

then we have the form in Equation (1) with $A = 2,\ B = 0,\ C = 1,$ and $q = 2.$ Then Equation (2) gives $\alpha = -1/2,\ \beta = 1,\ k = 1,$ and $p = 1/2,$ so by Equation (3) the general solution is

$$\begin{aligned} y(x) &= x^{-1/2}\left[c_1J_{1/2}(x) + c_1J_{-1/2}(x)\right] \\ &= x^{-1/2}\left[c_1\sqrt{\frac{2}{\pi x}}\cos x + c_2\sqrt{\frac{2}{\pi x}}\sin x\right] = \frac{1}{x}\left(a_1\cos x + a_2\sin x\right), \end{aligned}$$

(with $a_i = c_i\sqrt{2/\pi}$) using Equations (19) in Section 3.5.

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15. The substitution

$$y = -\frac{u'}{u}, \qquad y' = \frac{(u')^2}{u^2} - \frac{u''}{u}$$

immediately transforms $y' = x^2 + y^2$ to $u'' + x^2 u = 0$. The equivalent equation

$$x^2 u'' + x^4 u = 0$$

is of the form in (1) with $A = B = 0$, $C = 1$, and $q = 4$. Equations (2) give $\alpha = 1/2$, $\beta = 2$, $k = 1/2$, and $p = 1/4$, so the general solution is

$$u(x) = x^{1/2}[c_1 J_{1/4}(x^2/2) + c_2 J_{-1/4}(x^2/2)].$$

To compute $u'(x)$, let $z = x^2/2$ so $x = 2^{1/2} z^{1/2}$. Then Equation (22) in Section 8.5 with $p = 1/4$ yields

$$\begin{aligned}\frac{d}{dx}\left(x^{1/2} J_{1/4}(x^2/2)\right) &= \frac{d}{dz}\left(2^{1/4} z^{1/4} J_{1/4}(z)\right)\cdot\frac{dz}{dx}\\ &= 2^{1/4} z^{1/4} J_{-3/4}(z)\cdot\frac{dz}{dx}\\ &= 2^{1/4}\cdot\frac{x^{1/2}}{2^{1/4}} J_{-3/4}(x^2/2)\cdot x = x^{3/2} J_{-3/4}(x^2/2).\end{aligned}$$

Similarly, Equation (23) in Section 8.5 with $p = -1/4$ yields

$$\frac{d}{dx}\left(x^{1/2} J_{-1/4}(x^2/2)\right) = \frac{d}{dz}\left(2^{1/4} z^{1/4} J_{-1/4}(z)\right)\cdot\frac{dz}{dx} = -x^{3/2} J_{3/4}(x^2/2).$$

Therefore

$$u'(x) = x^{3/2}[c_1 J_{-3/4}(x^2/2) - c_2 J_{3/4}(x^2/2)].$$

It follows finally that the general solution of the Riccati equation $y' = x^2 + y^2$ is

$$y(x) = -\frac{u'}{u} = x\cdot\frac{J_{3/4}(\tfrac{1}{2}x^2) - c\,J_{-3/4}(\tfrac{1}{2}x^2)}{c\,J_{1/4}(\tfrac{1}{2}x^2) + J_{-1/4}(\tfrac{1}{2}x^2)}$$

where the arbitrary constant is $c = c_1/c_2$.

16. Substitution of the series expressions for the Bessel functions in the formula for $y(x)$ in Problem 15 yields

$$y(x) = x\cdot\frac{A\left(\tfrac{1}{2}x^2\right)^{3/4}(1+\cdots) - c\,B\left(\tfrac{1}{2}x^2\right)^{-3/4}(1+\cdots)}{c\,C\left(\tfrac{1}{2}x^2\right)^{1/4}(1+\cdots) + D\left(\tfrac{1}{2}x^2\right)^{-1/4}(1+\cdots)}$$

where each pair of parentheses encloses a power series in x with constant term 1, and

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$$A = 2^{-3/4}/\Gamma(7/4) \qquad B = 2^{3/4}/\Gamma(1/4)$$
$$C = 2^{-1/4}/\Gamma(5/4) \qquad D = 2^{1/4}/\Gamma(3/4).$$

Multiplication of numerator and denominator by $x^{1/2}$ and a bit of simplification gives

$$y(x) = \frac{2^{-3/4}Ax^3(1+\cdots) - 2^{3/4}cB(1+\cdots)}{2^{-1/4}cCx(1+\cdots) + 2^{1/4}D(1+\cdots)}.$$

It now follows that

$$y(0) = \frac{-2^{3/4}cB}{2^{1/4}D} = \frac{-2^{1/2}\left(2^{3/4}/\Gamma(1/4)\right)}{2^{1/4}/\Gamma(3/4)} = -2c\cdot\frac{\Gamma(3/4)}{\Gamma(1/4)}. \qquad (*)$$

(a) If $y(0) = 0$ then (*) gives $c = 0$ in the general solution formula of Problem 15.

(b) If $y(0) = 1$ then (*) gives $c = -\Gamma(1/4)/2\Gamma(3/4)$. More generally, (*) yields the formula

$$y(x) = x\cdot\frac{2\Gamma\left(\frac{3}{4}\right)J_{3/4}(\frac{1}{2}x^2) + y_0\,\Gamma\left(\frac{1}{4}\right)J_{-3/4}(\frac{1}{2}x^2)}{2\Gamma\left(\frac{3}{4}\right)J_{-1/4}(\frac{1}{2}x^2) - y_0\,\Gamma\left(\frac{1}{4}\right)J_{1/4}(\frac{1}{2}x^2)}$$

for the solution of the initial value problem

$$y' = x^2 + y^2, \qquad y(0) = y_0.$$

17. If we write the equation $x^4y'' + \gamma^2 y = 0$ in the form

$$x^2y'' + \gamma^2 x^{-2} y = 0,$$

then we see that it is of the form in Equation (3) of this section with $A = B = 0$, $C = \gamma^2$, and $q = -2$. Then Equations (5) give $\alpha = 1/2$, $\beta = -1$, $k = \gamma$, and $p = -1/2$, so the theorem yields the general solution

$$y(x) = x^{1/2}[c_1J_{1/2}(\gamma/x) + c_2J_{-1/2}(\gamma/x)] = x[A\cos(\gamma/x) + B\sin(\gamma/x)],$$

using Equations (19) in Section 8.5 for $J_{1/2}(x)$ and $J_{-1/2}(x)$. With a and b both nonzero, the initial conditions $y(a) = y(b) = 0$ yield the equations

$$A\cos(\gamma/a) + B\sin(\gamma/a) = 0$$
$$A\cos(\gamma/b) + B\sin(\gamma/b) = 0.$$

These equations have a nontrivial solution for A and B only if the coefficient determinant

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$$\begin{aligned}\Delta &= \sin(\gamma/b)\cos(\gamma/a) - \sin(\gamma/a)\cos(\gamma/b) \\ &= \sin(\gamma/b - \gamma/a) = \sin(\gamma L/ab)\end{aligned}$$

is nonzero. Hence $\gamma L/ab$ must be an integral multiple $n\pi$ of π, and then the nth buckling force is

$$P_n = \frac{EI_0\gamma_n^2}{b^4} = \frac{EI_0}{b^4}\left(\frac{n\pi ab}{L}\right)^2 = EI_0\left(\frac{n\pi}{L}\right)^2\left(\frac{a}{b}\right)^2.$$

18. The substitution $L = a + bt$ in $L\theta'' + 2L'\theta' + g\theta = 0$ yields the transformed equation

$$L^2\theta''(L) + 2L\theta'(L) + (g/b^2)L\theta = 0$$

with independent variable L that is of the form in (1) with $A = 2$, $B = 0$, $q = 1$, and $C = g/b^2$. Hence

$$\alpha = -1/2, \quad \beta = 1/2, \quad k = 2g^{1/2}/b, \quad \text{and} \quad p = 1,$$

so

$$\theta(L) = \frac{1}{\sqrt{L}}\left[AJ_1\left(\frac{2}{b}\sqrt{gL}\right) + BY_1\left(\frac{2}{b}\sqrt{gL}\right)\right].$$

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