CHAPTER 9

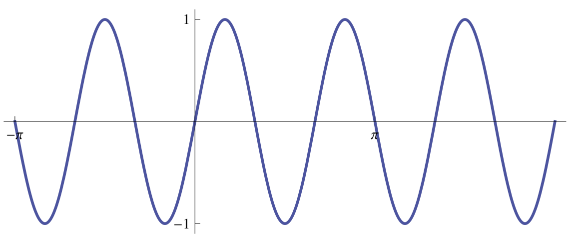
FOURIER SERIES METHODS AND PARTIAL DIFFERENTIAL EQUATIONS

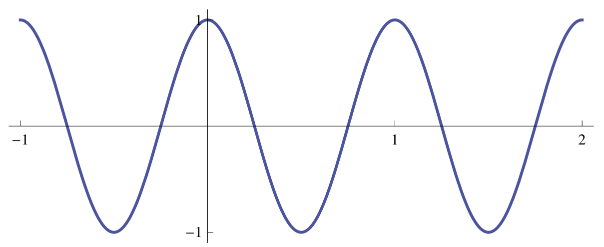
SECTION 9.1

PERIODIC FUNCTIONS AND TRIGONOMETRIC SERIES

The basic trigonometric functions $\cos(t)$ and $\sin(t)$ have period $P = 2\pi$, so the sine or cosine of ωt (as in Problems 1–4) completes its first period when $\omega t = 2\pi$; hence $P = 2\pi/\omega$.

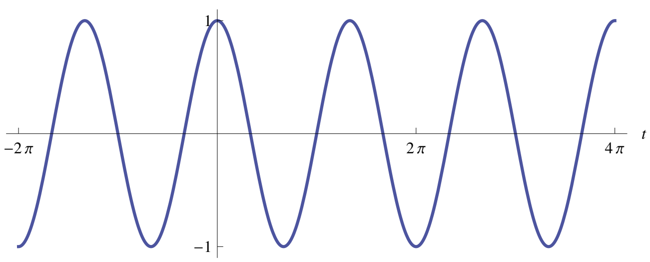
1. Smallest period $P = 2\pi/3$ (left-hand figure below)

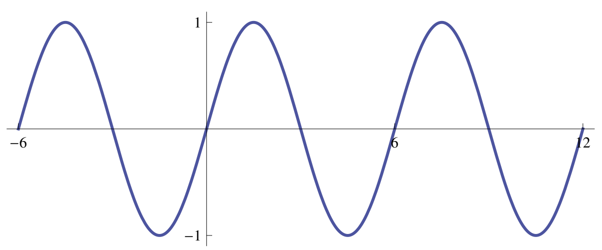




2. Smallest period $P = 1$ (right-hand figure above)

3. Smallest period $P = 4\pi/3$ (left-hand figure below)



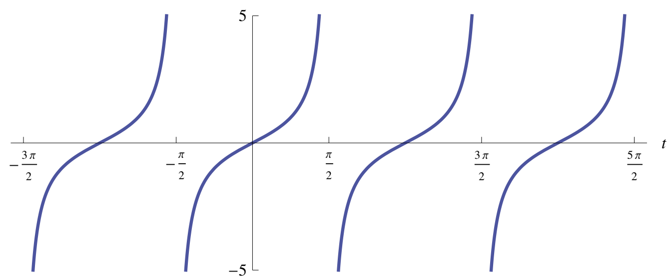


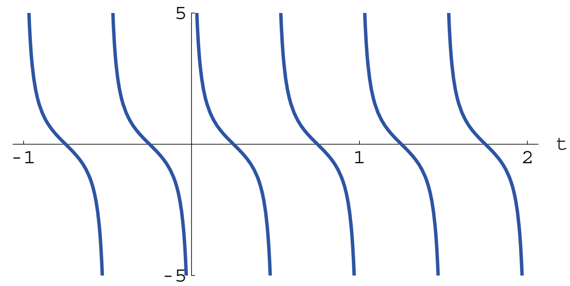
4. Smallest period $P = 6$ (right-hand figure above)

However, the basic tangent and cotangent functions have period π (instead of 2π), so $P = \pi/\omega$ in Problems 5 and 6.

5. Smallest period $P = \pi$; see the left-hand figure at the top of the next page.

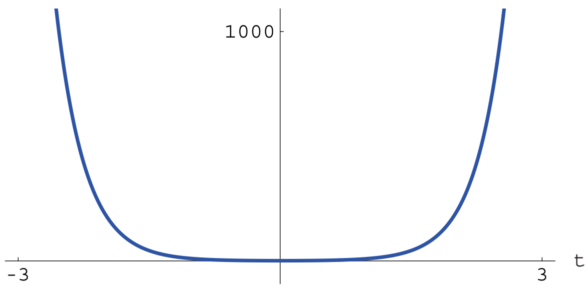
6. Smallest period $P = 1/2$; see the right-hand figure at the top of the next page.

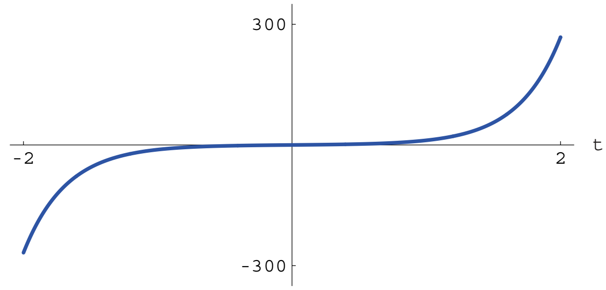




The hyperbolic sine and cosine functions of Problems 7 and 8 are steadily increasing (for $t > 0$), and hence are not periodic.

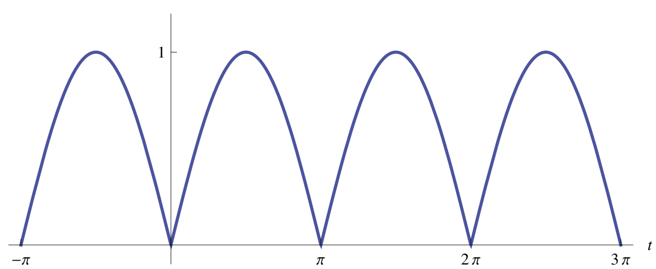
7. Not periodic (left-hand figure below)

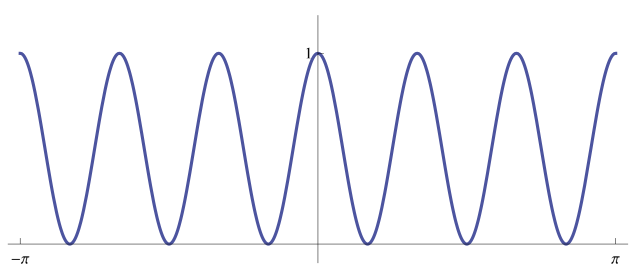




8. Not periodic (right-hand figure above)

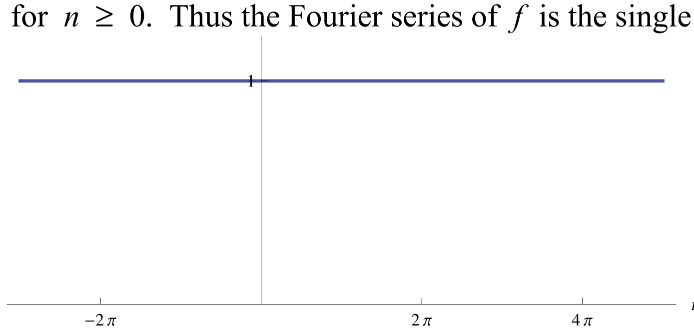
9. Smallest period $P = \pi$ (left-hand figure below)





10. Smallest period $P = \pi/3$ (right-hand figure above)

11. With $f(t) = 1$ the integral formulas of Eqs. (16) and (17) in the text give $a_0 = 2$ and $a_n = b_n = 0$ for $n \geq 0$. Thus the Fourier series of f is the single term series $f(t) = 1$.



In Problems 12–13, 18–19, 23, and 26 the function $f(t)$ is defined by one formula on the interval $(-\pi, 0)$ and by another formula on $(0, \pi)$. The coefficient integrals must therefore be split accordingly, and the appropriate formula substituted in each integral:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(t)\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\,dt,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(t)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos nt\,dt, \qquad (n > 0)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin nt\,dt.$$

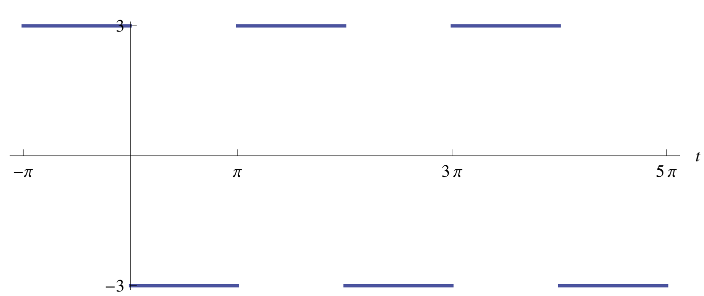
12. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} (+3)\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-3)\,dt = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (+3)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-3)\cos nt\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (+3)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-3)\sin nt\,dt =$$

$$= \frac{6}{n\pi}\left[\cos n\pi - 1\right] = \begin{cases} 0 & \text{for } n \text{ even} \\ -12/n\pi & \text{for } n \text{ odd} \end{cases}$$

$$f(t) \sim -\frac{12}{\pi}\left[\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \cdots\right]$$



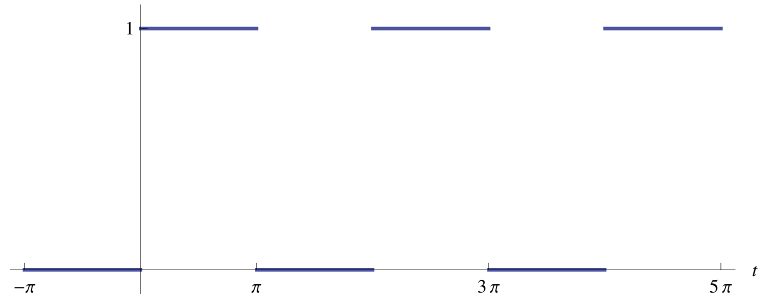
13. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} (0)\,dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\,dt = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (0)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\cos nt\,dt = \frac{\sin n\pi}{n\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (0)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\sin nt\,dt =$$

$$= \frac{1-\cos n\pi}{n\pi} = \begin{cases} 0 & \text{for } n \text{ even} \\ 2/n\pi & \text{for } n \text{ odd} \end{cases}$$

$$f(t) \sim \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \cdots\right] \qquad \text{(figure below)}$$



14.

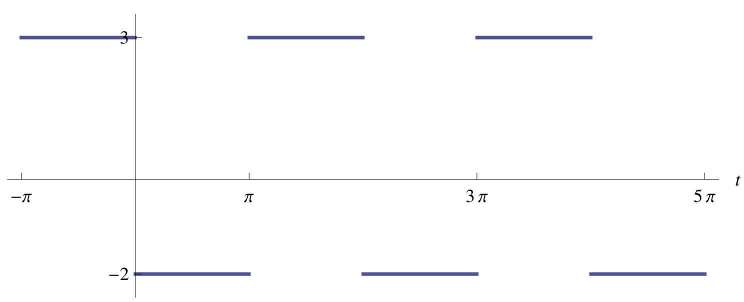
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} (3)\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-2)\,dt = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} (3)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-2)\cos nt\,dt = \frac{\sin n\pi}{n\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} (3)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-2)\sin nt\,dt =$$

$$= \frac{5(\cos n\pi - 1)}{n\pi} = \begin{cases} 0 & \text{for } n \text{ even} \\ -10/n\pi & \text{for } n \text{ odd} \end{cases}$$

$$f(t) \sim \frac{1}{2} - \frac{10}{\pi}\left[\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \cdots\right] \qquad \text{(figure below)}$$

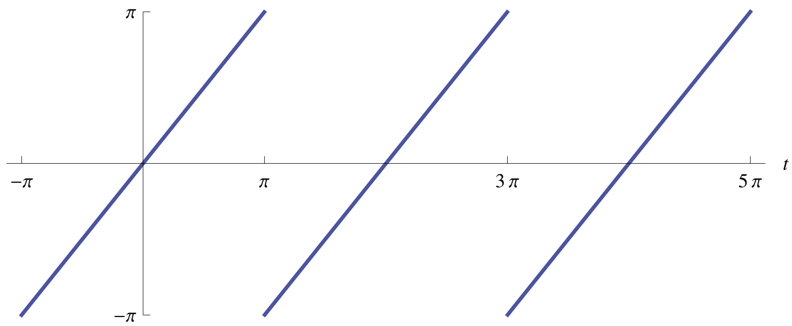


15.

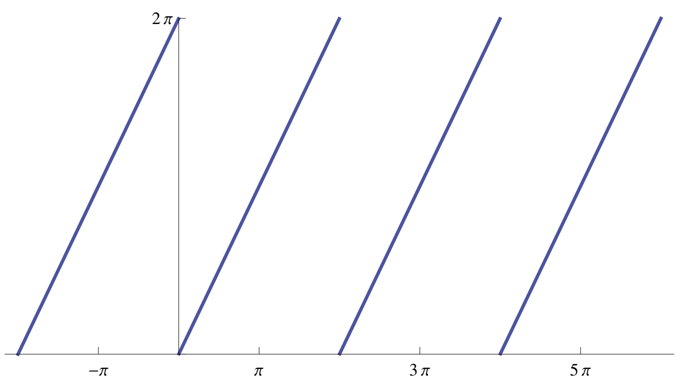
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = 0, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{2\sin n\pi - 2n\pi\cos n\pi}{n^2\pi} = \begin{cases} -2/n & \text{for } n \text{ even} \\ +2/n & \text{for } n \text{ odd} \end{cases}$$

$$f(t) \sim 2\left[\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \cdots\right] \qquad \text{(figure at top of next page)}$$

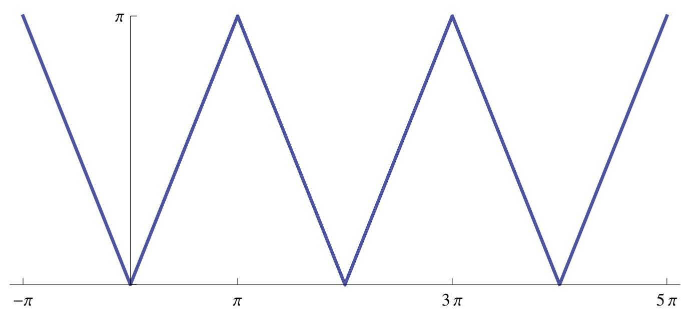


16.
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t\,dt = 2\pi$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos nt\,dt = \frac{\cos 2n\pi - 2n\pi\sin 2n\pi - 1}{n^2\pi} = 0$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin nt\,dt = \frac{2\sin 2n\pi - 2n\pi\cos 2n\pi}{n^2\pi} = -\frac{2}{n}$$
$$f(t) \sim \pi - 2\left[\frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right]$$ (figure below)



17.
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-t)\,dt + \frac{1}{\pi}\int_0^{\pi}(t)\,dt = \pi$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-t)\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}(t)\cos nt\,dt$$
$$= \frac{2(\cos n\pi + n\pi\sin n\pi - 1)}{n^2\pi} = \begin{cases} 0 \text{ for } n \text{ even} \\ -4/n^2\pi \text{ for } n \text{ odd}\end{cases}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-t)\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi}(t)\sin nt\,dt = 0$$
$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos t}{1} + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \frac{\cos 7t}{49} + \cdots\right]$$

See the figure at the top of the next page.



18.

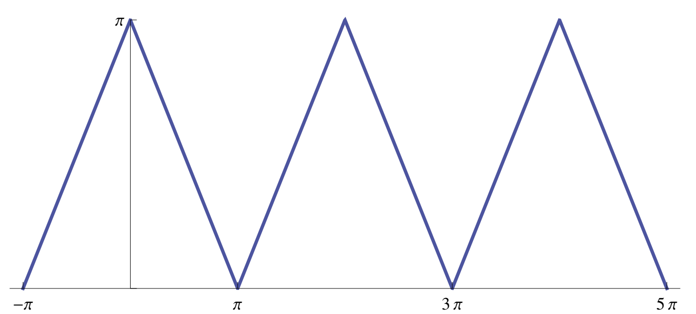
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\,dt + \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\,dt = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\cos nt\,dt$$

$$= \frac{2(1-\cos n\pi)}{n^2\pi} = \begin{cases} 0 \text{ for } n \text{ even} \\ 4/n^2\pi \text{ for } n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\sin nt\,dt = 0$$

$$f(t) \sim \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos t}{1} + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \frac{\cos 7t}{49} + \cdots\right] \qquad \text{(figure below)}$$



19.

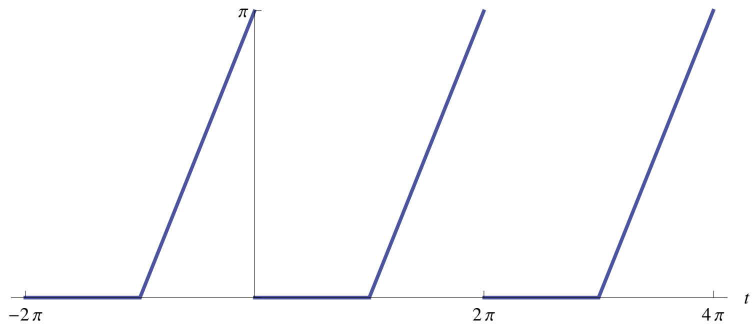
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\,dt + \frac{1}{\pi}\int_{0}^{\pi}(0)\,dt = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(0)\cos nt\,dt$$

$$= \frac{1-\cos n\pi}{n^2\pi} = \begin{cases} 0 \text{ for } n \text{ even} \\ 2/n^2\pi \text{ for } n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(0)\sin nt\,dt = \frac{\sin n\pi - n\pi}{n^2\pi} = -\frac{1}{n}$$

$$f(t) \sim \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos t}{1} + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \frac{\cos 7t}{49} + \cdots\right] - \left[\frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right]$$



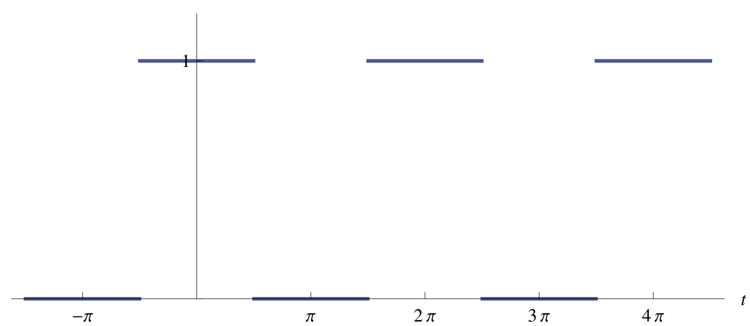
20.

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dt = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nt\,dt = \frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{for } n \text{ even} \\ +1 & \text{for } n = 1, 5, \cdots \\ -1 & \text{for } n = 3, 7, \cdots \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nt\,dt = 0$$

$$f(t) \sim \frac{1}{2} + \frac{2}{\pi}\left[\frac{\cos t}{1} - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \frac{\cos 7t}{7} + \cdots\right]$$ (figure below)



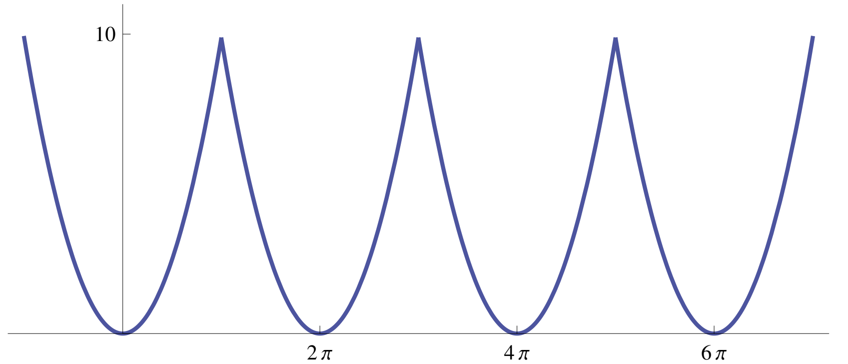
21.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\,dt = \frac{4n\pi\cos n\pi - 2(n^2\pi^2 - 2)\sin n\pi}{n^3\pi} = \begin{cases} +4/n^2 & \text{for } n \text{ even} \\ -4/n^2 & \text{for } n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin nt\,dt = 0$$

$$f(t) \sim \frac{\pi^2}{3} - 4\left[\frac{\cos t}{1} - \frac{\cos 2t}{4} + \frac{\cos 3t}{9} - \frac{\cos 4t}{16} + \cdots\right] \qquad \text{(figure below)}$$



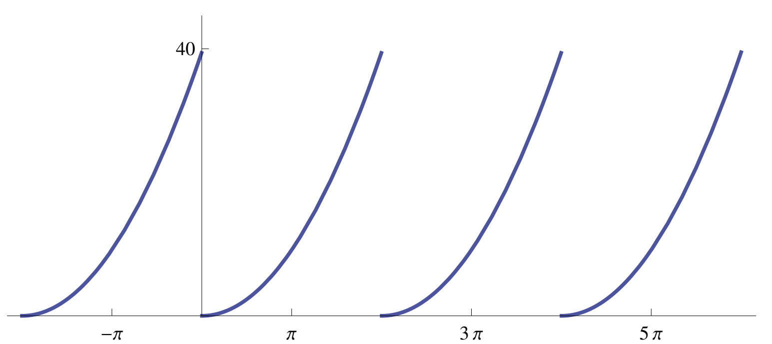
22.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt\,dt = \frac{4n\pi\cos 2n\pi + 2(n^2\pi^2 - 1)\sin 2n\pi}{n^3\pi} = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt = \frac{(2 - 4n^2\pi^2)\cos 2n\pi + 4n\pi\sin 2n\pi - 2}{n^3\pi} = -\frac{4\pi}{n}$$

$$f(t) \sim \frac{4\pi^2}{3} + 4\left[\frac{\cos t}{1} + \frac{\cos 2t}{4} + \frac{\cos 3t}{9} + \frac{\cos 4t}{16} + \cdots\right] - 4\pi\left[\frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right] \qquad \text{(figure below)}$$



23.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} t^2\,dt = \frac{\pi^2}{3}$$

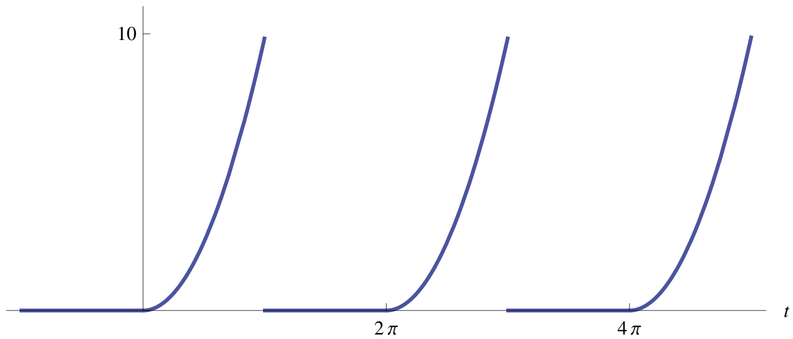
$$a_n = \frac{1}{\pi}\int_0^{\pi} t^2 \cos nt\,dt = \frac{2n\pi\cos n\pi + (n^2\pi^2 - 2)\sin n\pi}{n^3\pi} = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt$$

$$= \frac{(2-n^2\pi^2)\cos n\pi + 2n\pi\sin n\pi - 2}{n^3\pi} = \begin{cases} -\pi/n & \text{for } n \text{ even} \\ (n^2\pi^2-4)/\pi n^3 & \text{for } n \text{ odd} \end{cases}$$

$$f(t) \sim \frac{\pi^2}{6} - 2\left[\frac{\cos t}{1} - \frac{\cos 2t}{4} + \frac{\cos 3t}{9} - \frac{\cos 4t}{16} + \cdots\right] \qquad \text{(figure below)}$$

$$+\pi\left[\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \cdots\right] - \frac{4}{\pi}\left[\frac{\sin t}{1} + \frac{\sin 3t}{27} + \frac{\sin 5t}{125} + \frac{\sin 7t}{343} + \cdots\right]$$



The trigonometric identities

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

are needed to evaluate the integrals that appear in Problems 24–26.

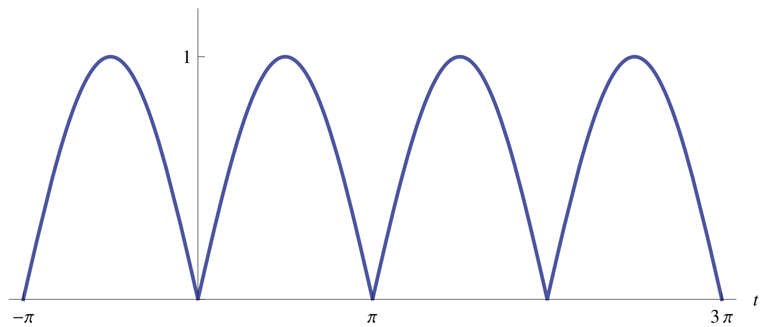
24. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin t)\,dt + \frac{1}{\pi}\int_0^{\pi}(\sin t)\,dt = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin t)\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}(\sin t)\cos nt\,dt$$

$$= \frac{2(1+\cos n\pi)}{\pi(1-n^2)} = \begin{cases} -4/\pi(n^2-1) & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin t)\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi}(\sin t)\sin nt\,dt = 0$$

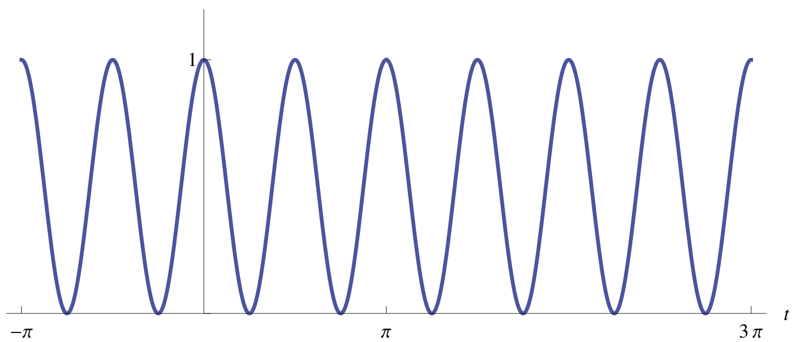
$$f(t) \sim \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2t}{1} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \frac{\cos 8t}{63} + \cdots\right] \qquad \text{(figure at top of next page)}$$



25. In order to evaluate the coefficient integrals in Eqs. (16) and (17) of the text we would need the trigonometric identity

$$\cos^2 2t = \frac{1}{2}(1+\cos 4t)$$

which, however, tells us in advance that the coefficients in the Fourier series of $f(t) = \cos^2 2t$ are given by $a_0 = 1$, $a_4 = 1/2$, $a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.



26.

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(\sin t)\,dt = \frac{2}{\pi}$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin t\cos t\,dt = 0$$

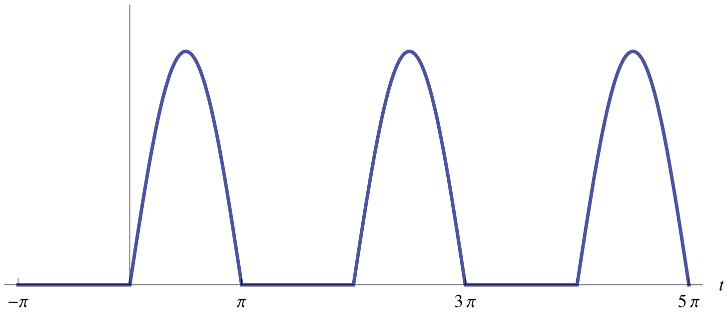
$$a_n = \frac{1}{\pi}\int_0^{\pi}(\sin t)\cos nt\,dt = \frac{1+\cos n\pi}{\pi(1-n^2)} = \begin{cases} -2/\pi(n^2-1) & \text{for } n \text{ even} \\ 0 & \text{for } n>1 \text{ odd} \end{cases}$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin^2 t\,dt = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\sin t)\sin nt\,dt = \frac{\sin n\pi}{\pi(1-n^2)} = 0 \text{ for } n>1$$

$$f(t) \sim \frac{1}{\pi}+\frac{1}{2}\sin t-\frac{2}{\pi}\left[\frac{\cos 2t}{1}+\frac{\cos 4t}{15}+\frac{\cos 6t}{35}+\frac{\cos 8t}{63}+\cdots\right]$$

Note that $f(t) = (\sin t + |\sin t|)/2$, so this answer agrees with the answer to Problem 24.



SECTION 9.2

GENERAL FOURIER SERIES AND CONVERGENCE

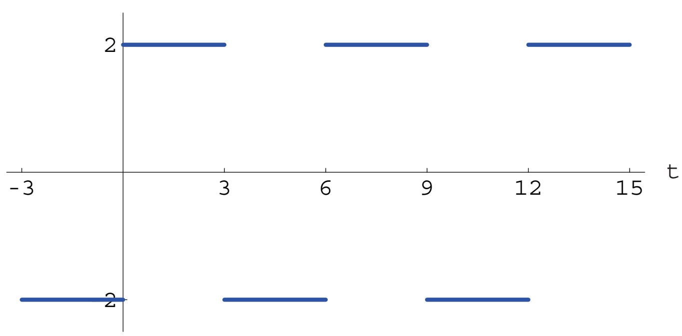
1.

$$a_0 = \frac{1}{3}\int_{-3}^{0}(-2)\,dt + \frac{1}{3}\int_0^3 (2)\,dt = 0$$

$$a_n = \frac{1}{3}\int_{-3}^{0}(-2)\cos\frac{n\pi t}{3}\,dt + \frac{1}{3}\int_0^3 (2)\cos\frac{n\pi t}{3}\,dt = 0$$

$$b_n = \frac{1}{3}\int_{-3}^{0}(-2)\sin\frac{n\pi t}{3}\,dt + \frac{1}{3}\int_0^3 (2)\sin\frac{n\pi t}{3}\,dt = \frac{4(1-\cos n\pi)}{n\pi} = \frac{4}{n\pi}\left[1-(-1)^n\right]$$

$$f(t) = \frac{8}{\pi}\left[\sin\frac{\pi t}{3} + \frac{1}{3}\sin\frac{3\pi t}{3} + \frac{1}{5}\sin\frac{5\pi t}{3} + \frac{1}{7}\sin\frac{7\pi t}{3} + \cdots\right] \qquad \text{(figure below)}$$

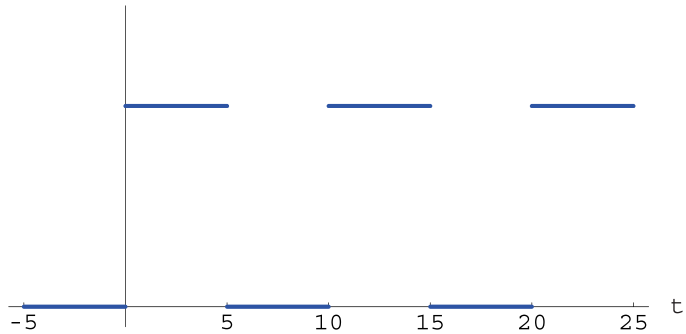


2.

$$a_0 = \frac{1}{5}\int_0^5 (1)\,dt = 1, \qquad a_n = \frac{1}{5}\int_0^5 (1)\cos\frac{n\pi t}{5}\,dt = \frac{\sin n\pi}{n\pi} = 0$$

$$b_n = \frac{1}{5}\int_0^5 (1)\sin\frac{n\pi t}{5}\,dt = \frac{1-\cos n\pi}{n\pi} = \frac{1-(-1)^n}{n\pi}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi t}{5} + \frac{1}{3}\sin\frac{3\pi t}{5} + \frac{1}{5}\sin\frac{5\pi t}{5} + \frac{1}{7}\sin\frac{7\pi t}{5} + \cdots\right] \qquad \text{(figure below)}$$

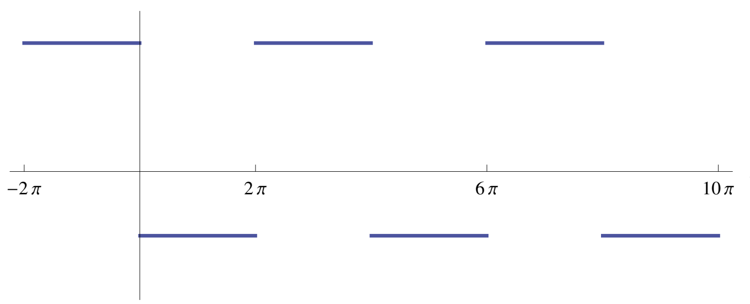


3.
$$a_0 = \frac{1}{2\pi}\int_{-2\pi}^{0}(2)\,dt + \frac{1}{2\pi}\int_{0}^{2\pi}(-1)\,dt = 1$$

$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{0}(2)\cos\frac{nt}{2}\,dt + \frac{1}{2\pi}\int_{0}^{2\pi}(-1)\cos\frac{nt}{2}\,dt = \frac{\sin n\pi}{n\pi} = 0$$

$$b_n = \frac{1}{2\pi}\int_{-2\pi}^{0}(2)\sin\frac{nt}{2}\,dt + \frac{1}{2\pi}\int_{0}^{2\pi}(-1)\sin\frac{nt}{2}\,dt = \frac{3(\cos n\pi - 1)}{n\pi} = \frac{3}{n\pi}\left[(-1)^n - 1\right]$$

$$f(t) = \frac{1}{2} - \frac{6}{\pi}\left[\sin\frac{t}{2} + \frac{1}{3}\sin\frac{3t}{2} + \frac{1}{5}\sin\frac{5t}{2} + \frac{1}{7}\sin\frac{7t}{2} + \cdots\right] \qquad \text{(figure below)}$$

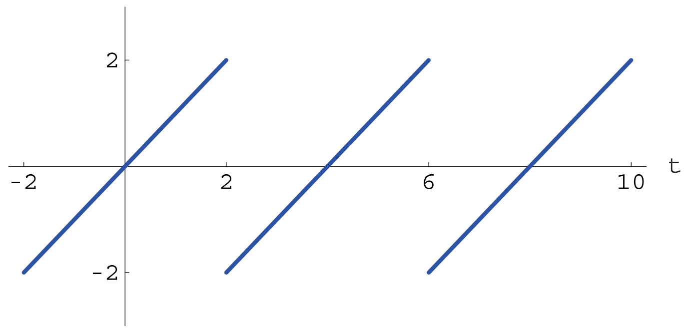


4.
$$a_0 = \frac{1}{2}\int_{-2}^{2} t\,dt = 0, \qquad a_n = \frac{1}{2}\int_{-2}^{2} t\cos\frac{n\pi t}{2}\,dt = 0$$

$$b_n = \frac{1}{2}\int_{-2}^{2} t\sin\frac{n\pi t}{2}\,dt = \frac{4(\sin n\pi - n\pi\cos n\pi)}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n\pi}$$

$$f(t) = \frac{4}{\pi}\left[\sin\frac{\pi t}{2} - \frac{1}{2}\sin\frac{2\pi t}{2} + \frac{1}{3}\sin\frac{3\pi t}{2} - \frac{1}{4}\sin\frac{4\pi t}{2} + \cdots\right]$$

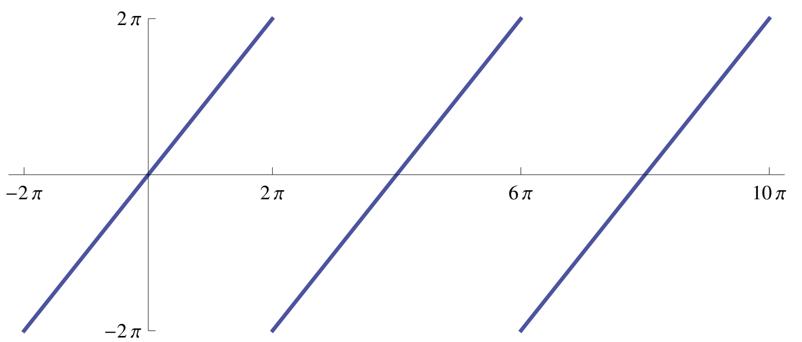
See figure at top of next page.



5. $a_0 = \dfrac{1}{2\pi}\int_{-2\pi}^{2\pi} t\,dt = 0,$ $\qquad a_n = \dfrac{1}{2\pi}\displaystyle\int_{-2\pi}^{2\pi} t\cos\dfrac{nt}{2}\,dt = 0$

$$b_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} t\sin\frac{nt}{2}\,dt = \frac{4(\sin n\pi - n\pi\cos n\pi)}{n^2\pi} = \frac{4(-1)^{n+1}}{n}$$

$$f(t) = 4\left[\sin\frac{t}{2} - \frac{1}{2}\sin\frac{2t}{2} + \frac{1}{3}\sin\frac{3t}{2} - \frac{1}{4}\sin\frac{4t}{2} + \cdots\right]$$ (figure below)

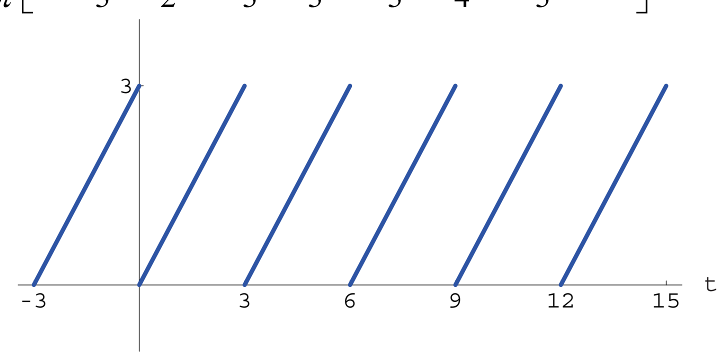


6. $a_0 = \dfrac{2}{3}\int_0^3 t\,dt = 3$

$$a_n = \frac{2}{3}\int_0^3 t\cos\frac{2n\pi t}{3}\,dt = \frac{3\left[\cos 2n\pi + 2n\pi\sin 2n\pi - 1\right]}{2n^2\pi^2} = 0$$

$$b_n = \frac{2}{3}\int_0^3 t\sin\frac{2n\pi t}{3}\,dt = \frac{3\sin 2n\pi - 6n\pi\cos 2n\pi}{2n^2\pi^2} = \frac{3(-1)^{n+1}}{n\pi}$$

$$f(t) = \frac{3}{2} - \frac{3}{\pi}\left[\sin\frac{2\pi t}{3} + \frac{1}{2}\sin\frac{4\pi t}{3} + \frac{1}{3}\sin\frac{6\pi t}{3} + \frac{1}{4}\sin\frac{8\pi t}{3} + \cdots\right]$$ (figure below)



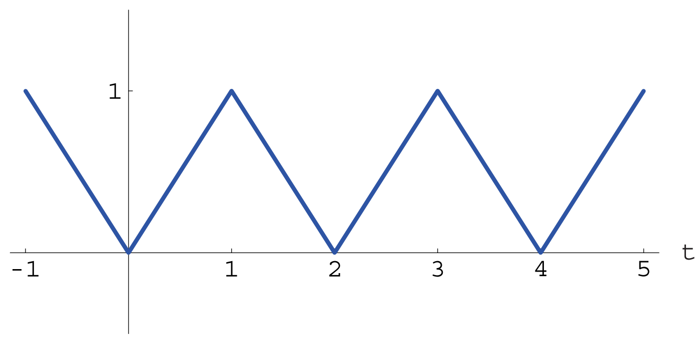
7.

$$a_0 = \int_{-1}^{0} (-t)\,dt + \int_0 (t)\,dt = 1$$

$$a_n = \int_{-1}^{0} (-t)\cos n\pi t\,dt + \int_0^1 t\cos n\pi t\,dt = \frac{2[\cos n\pi + n\pi\sin n\pi - 1]}{n^2\pi^2} = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \int_{-1}^{0} (-t)\sin n\pi t\,dt + \int_0^1 t\sin n\pi t\,dt = 0$$

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\left[\cos\pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \frac{1}{49}\cos 7\pi t + \cdots\right] \qquad \text{(figure below)}$$



8.

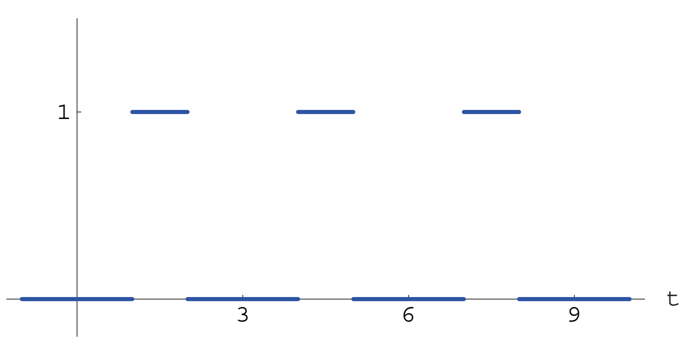
$$a_0 = \frac{2}{3}\int_1^2 1\,dt = \frac{2}{3}$$

$$a_n = \frac{2}{3}\int_1^2 \cos\frac{2n\pi t}{3}\,dt = \frac{1}{n\pi}\left[\sin\frac{4n\pi}{3} - \sin\frac{2n\pi}{3}\right]$$

$$b_n = \frac{2}{3}\int_1^2 \sin\frac{2n\pi t}{3}\,dt = \frac{1}{n\pi}\left[\cos\frac{2n\pi}{3} - \cos\frac{4n\pi}{3}\right]$$

Analyzing separately the cases $n = 3k$, $n = 3k+1$, and $n = 3k+2$, we find that $a_{3k} = 0$, $a_{3k+1} = -\sqrt{3}/\pi n$, $a_{3k+2} = +\sqrt{3}/\pi n$, and that $b_n = 0$ for all n. Hence the Fourier series of $f(t)$ is

$$f(t) = \frac{1}{3} - \frac{\sqrt{3}}{\pi}\left[\cos\frac{2\pi t}{3} - \frac{1}{2}\cos\frac{4\pi t}{3} + \frac{1}{4}\cos\frac{8\pi t}{3} - \frac{1}{5}\cos\frac{10\pi t}{3} + \frac{1}{7}\cos\frac{14\pi t}{3} - \cdots\right].$$

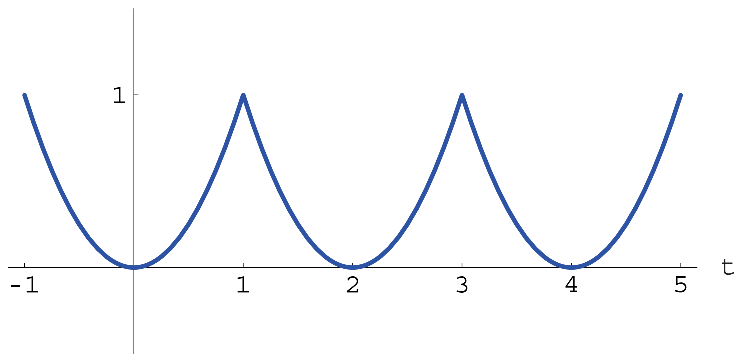


9. $a_0 = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$

$$a_n = \int_{-1}^{1} t^2 \cos n\pi t\,dt = \frac{4n\pi\cos n\pi + 2(n^2\pi^2 - 2)\sin n\pi}{n^3\pi^3} = \frac{4(-1)^n}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} t^2 \sin n\pi t\,dt = 0$$

$$f(t) = \frac{1}{3} - \frac{4}{\pi^2}\left[\cos\pi t - \frac{1}{4}\cos 2\pi t + \frac{1}{9}\cos 3\pi t - \frac{1}{16}\cos 4\pi t + \cdots\right] \qquad \text{(figure below)}$$



10. $a_0 = \frac{1}{2}\int_0^2 t^2\,dt = \frac{4}{3}$

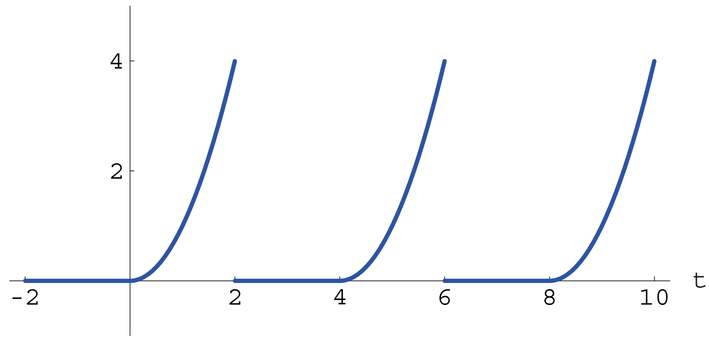
$$a_n = \frac{1}{2}\int_0^2 t^2 \cos\frac{n\pi t}{2}\,dt = \frac{8n\pi\cos n\pi + (n^2\pi^2 - 2)\sin n\pi}{n^3\pi^3} = \frac{8(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{2}\int_0^2 t^2 \sin\frac{n\pi t}{2}\,dt$$

$$= -\frac{4\left[(n^2\pi^2 - 2)\cos n\pi - 2n\pi\sin n\pi + 2\right]}{n^3\pi^3} = \begin{cases} -4/n\pi & \text{for } n \text{ even} \\ +4/n\pi - 16/n^3\pi^3 & \text{for } n \text{ odd} \end{cases}$$

$$f(t) = \frac{2}{3} - \frac{8}{\pi^2}\left[\cos\frac{\pi t}{2} - \frac{1}{4}\cos\frac{2\pi t}{2} + \frac{1}{9}\cos\frac{3\pi t}{2} - \frac{1}{16}\cos\frac{4\pi t}{2} + \cdots\right]$$
$$+ \frac{4}{\pi}\left[\sin\frac{\pi t}{2} - \frac{1}{2}\sin\frac{2\pi t}{2} + \frac{1}{3}\sin\frac{3\pi t}{2} - \cdots\right]$$
$$- \frac{16}{\pi^3}\left[\sin\frac{\pi t}{2} + \frac{1}{27}\sin\frac{3\pi t}{2} + \frac{1}{125}\sin\frac{5\pi t}{2} + \cdots\right]$$

See the figure at the top of the next page.



To calculate the Fourier coefficients in Problems 11–14 we use the trigonometric identities for sin A cos B and sin A sin B that are listed above in Section 9.1 (prior to Problems 24–26 there).

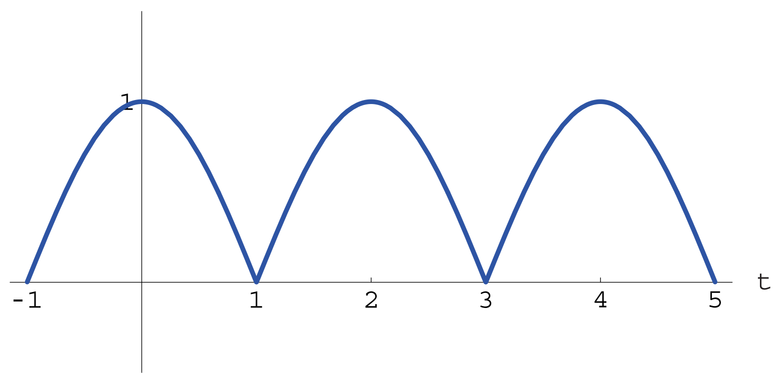
11.

$$a_0 = \int_{-1}^{1} \cos\frac{\pi t}{2}\,dt = \frac{4}{\pi}$$

$$a_n = \int_{-1}^{1} \cos\frac{\pi t}{2}\cos n\pi t\,dt = -\frac{4\cos n\pi}{\pi(4n^2-1)} = \frac{4(-1)^{n+1}}{\pi(4n^2-1)}$$

$$b_n = \int_{-1}^{1} \cos\frac{\pi t}{2}\sin n\pi t\,dt = 0$$

$$f(t) = \frac{2}{\pi}+\frac{4}{\pi}\left[\frac{1}{3}\cos\pi t-\frac{1}{15}\cos 2\pi t+\frac{1}{35}\cos 3\pi t-\frac{1}{63}\cos 4\pi t+\cdots\right] \qquad \text{(figure below)}$$



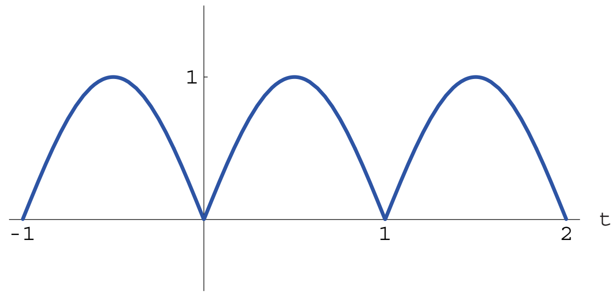
12.

$$a_0 = 2\int_{0}^{1} \sin\pi t\,dt = \frac{4}{\pi}$$

$$a_n = 2\int_{0}^{1} \sin\pi t\cos 2n\pi t\,dt = -\frac{4\cos^2 n\pi}{\pi(4n^2-1)} = -\frac{4}{\pi(4n^2-1)}$$

$$b_n = 2\int_{0}^{1} \sin\pi t\sin 2n\pi t\,dt = -\frac{4\cos n\pi\sin n\pi}{\pi(4n^2-1)} = 0$$

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2\pi t + \frac{1}{15}\cos 4\pi t + \frac{1}{35}\cos 6\pi t + \frac{1}{63}\cos 8\pi t + \cdots\right]$$



13.

$$a_0 = \int_0^1 \sin \pi t\, dt = \frac{2}{\pi}$$

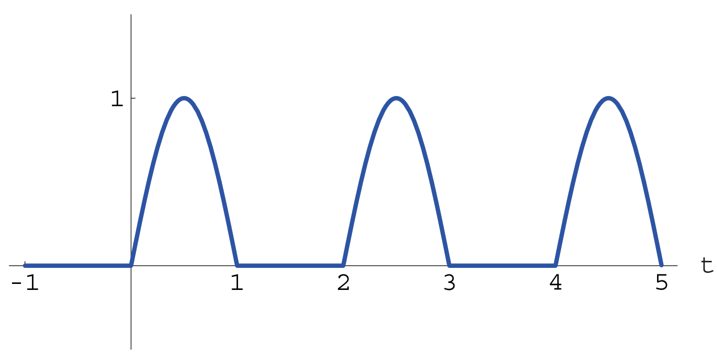
$$a_n = \int_0^1 \sin \pi t \cos n\pi t\, dt = -\frac{1+\cos n\pi}{\pi(n^2-1)} = -\frac{1+(-1)^n}{\pi(n^2-1)} \text{ for } n>1$$

$$a_1 = \int_0^1 \sin \pi t \cos \pi t\, dt = 0$$

$$b_n = \int_0^1 \sin \pi t \sin n\pi t\, dt = -\frac{\sin n\pi}{\pi(n^2-1)} = 0 \text{ for } n>1$$

$$b_1 = \int_0^1 \sin^2 \pi t\, dt = \frac{1}{2}$$

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin \pi t - \frac{2}{\pi}\left[\frac{1}{3}\cos 2\pi t + \frac{1}{15}\cos 4\pi t + \frac{1}{35}\cos 6\pi t + \frac{1}{63}\cos 8\pi t + \cdots\right]$$



14.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \sin t\, dt = 0$$

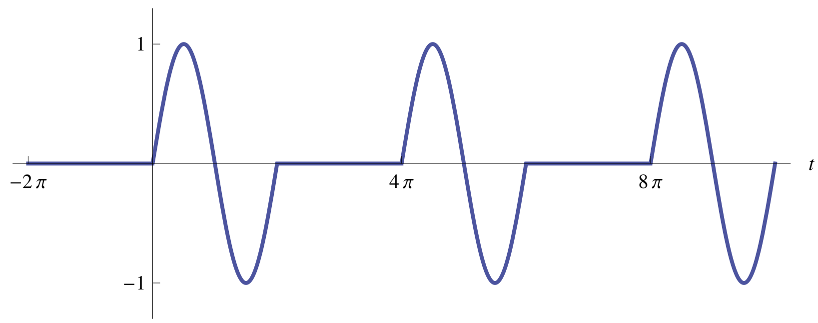
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \sin t \cos \frac{nt}{2}\, dt = \frac{2(\cos n\pi - 1)}{\pi(n^2-4)} = \frac{2\left[(-1)^n - 1\right]}{\pi(n^2-4)} \text{ for } n \neq 2$$

$$a_2 = \frac{1}{2\pi}\int_0^{2\pi} \sin t \cos t\, dt = 0$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \sin t \sin\frac{nt}{2}\,dt = \frac{2\sin n\pi}{\pi(n^2-4)} = 0 \text{ for } n \neq 2$$

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 t\,dt = \frac{1}{2}$$

$$f(t) = \frac{1}{2}\sin t + \frac{4}{\pi}\left[\frac{1}{3}\cos\frac{t}{2} - \frac{1}{5}\cos\frac{3t}{2} - \frac{1}{21}\cos\frac{5t}{2} - \frac{1}{45}\cos\frac{7t}{2} - \cdots\right] \qquad \text{(figure below)}$$

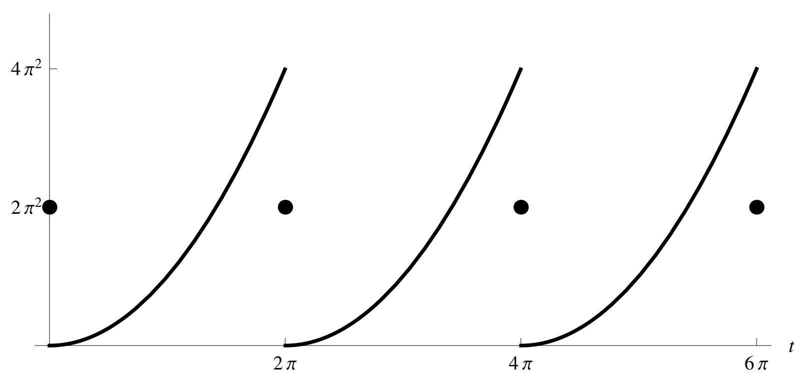


15. **(a)** $$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt\,dt = \frac{4n\pi\cos 2n\pi + 2(2n^2\pi^2 - 1)\sin 2n\pi}{\pi n^3} = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt = \frac{(2-4n^2\pi^2)\cos 2n\pi + 4n\pi\sin 2n\pi - 2}{\pi n^3} = -\frac{4\pi}{n}$$

$$f(t) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nt}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nt}{n} \qquad \text{(figure below)}$$



(b) If we substitute $t = 0$ in the Fourier series of part (a) and note that $f(0) = \frac{1}{2}\left[f(0-) + f(0+)\right] = \frac{1}{2}\left[(2\pi)^2 + (0)^2\right] = 2\pi^2$, we get

$$2\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}, \quad \text{so} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

When we substitute $t = \pi$ and $f(\pi) = \pi^2$ in the series of part (a) we get

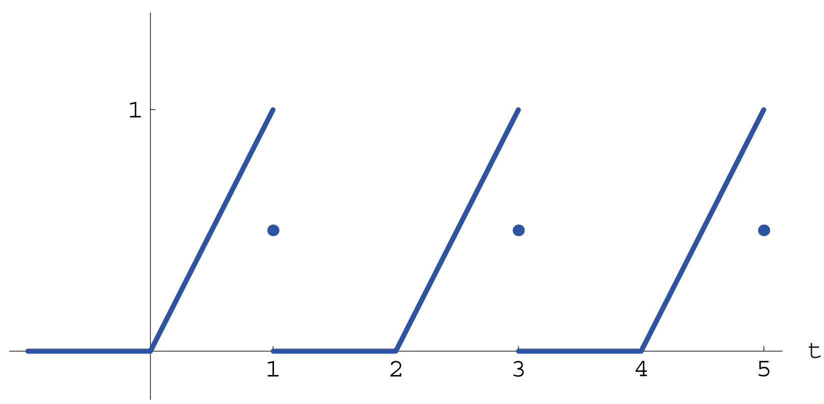
$$\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}, \quad \text{so} \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

16. **(a)** $a_0 = \int_0^1 t\,dt = \frac{1}{2}$

$$a_n = \int_0^1 t\cos n\pi t\,dt = \frac{\cos n\pi + n\pi\sin n\pi - 1}{n^2\pi^2} = \frac{(-1)^n - 1}{n^2\pi^2}$$

$$b_n = \int_0^1 t\sin n\pi t\,dt = \frac{\sin n\pi - n\pi\cos n\pi}{n^2\pi^2} = \frac{(-1)^{n+1}}{n\pi}$$

$$f(t) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{n\text{ odd}}\frac{\cos\, n\pi t}{n^2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi t}{n}$$ (figure below)



(b) Substitution of $t = 0,\ f(t) = 0$ in this series immediately gives $\sum_{n\text{ odd}}\frac{1}{n^2} = \frac{\pi^2}{8}$.

17. **(a)** $a_0 = \int_0^2 t\,dt = 2$

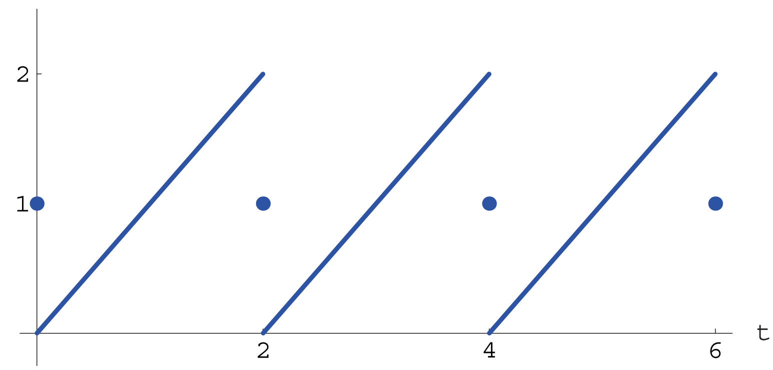
$$a_n = \int_0^2 t\cos n\pi t\,dt = \frac{\cos 2n\pi + 2n\pi\sin 2n\pi - 1}{n^2\pi^2} = 0$$

$$b_n = \int_0^2 t\sin n\pi t\,dt = \frac{\sin 2n\pi - 2n\pi\cos 2n\pi}{n^2\pi^2} = -\frac{2}{n\pi}$$

$$f(t) = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi t}{n}$$ (see figure on next page)

(b) Substitution of $t = 1/2,\ f(t) = 1/2$ in this series gives

$$\frac{1}{2} = 1 - \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \quad \text{so} \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$



The most efficient approach to Problems 18 and 20 is to derive first the expansions

$$t = \pi - 2\left[\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right],$$

$$t^2 = \frac{4\pi^2}{3} + 4\left[\cos t + \frac{\cos 2t}{4} + \frac{\cos 3t}{9} + \frac{\cos 4t}{16} + \cdots\right]$$
$$- 4\pi\left[\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right].$$

for $0 < t < 2\pi$, as the Fourier series of the functions $f(t)$ and $g(t)$ of period 2π defined for $0 < t < 2\pi$ by $f(t) = t$ and $g(t) = t^2$. The first series above yields the series in Problem 18, and a combination of the two yields the series in Problem 20.

The expansions in Problems 19 and 21 are valid on the interval $-\pi < t < \pi$ rather than the interval $0 < t < 2\pi$. When we calculate the Fourier series of the functions $f(t)$ and $g(t)$ of period 2π defined for $-\pi < t < \pi$ by $f(t) = t$ and $g(t) = t^2$, we find that

$$t = 2\left[\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \cdots\right],$$

$$t^2 = \frac{\pi^2}{3} - 4\left[\cos t - \frac{\cos 2t}{4} + \frac{\cos 3t}{9} - \frac{\cos 4t}{16} + \cdots\right]$$

if $-\pi < t < \pi$.

18. First we derive the the Fourier series of the function $f(t)$ of period 2π defined for $0 < t < 2\pi$ by $f(t) = t$.

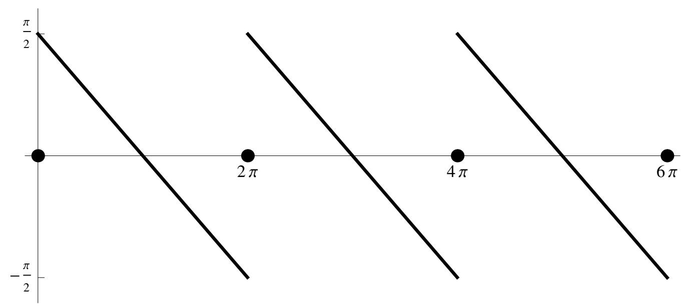
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t\,dt = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos nt\,dt = \frac{\cos 2n\pi + 2n\pi\sin 2n\pi - 1}{n^2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin nt\,dt = \frac{\sin 2n\pi - 2n\pi\cos 2n\pi}{n^2\pi} = -\frac{2}{n}$$

$$f(t) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

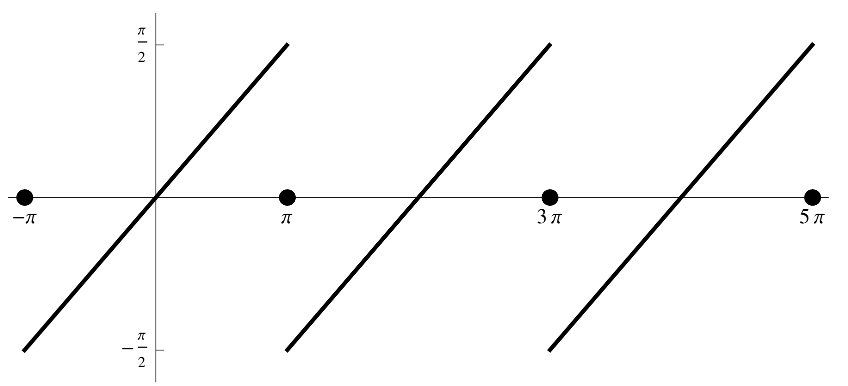
Hence $\dfrac{\pi - t}{2} = \dfrac{1}{2}\left[\pi - f(t)\right] = \displaystyle\sum_{n=1}^{\infty}\frac{\sin nt}{n}$ for $0 < t < 2\pi$ (figure below).



19. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\frac{t}{2}\,dt = 0,\qquad a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\frac{t}{2}\cos nt\,dt = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t}{2}\sin nt\,dt = \frac{\sin n\pi - n\pi\cos n\pi}{n^2\pi} = \frac{(-1)^{n+1}}{n}$$

$$\frac{t}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nt}{n} \qquad (-\pi < t < \pi) \qquad \text{(figure below)}$$



20. First we derive the the Fourier series of the function $g(t)$ of period 2π defined for $0 < t < 2\pi$ by $g(t) = t^2$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt\,dt = \frac{4n\pi\cos 2n\pi + 2(2n^2\pi^2-1)\sin 2n\pi}{n^3\pi} = \frac{4}{n^2}$$

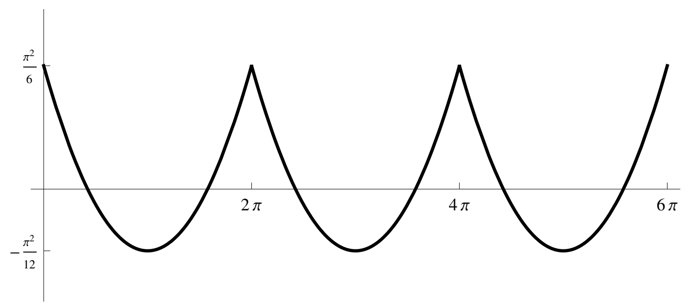
$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt = \frac{(2-4n^2\pi^2)\cos 2n\pi + 4n\pi\sin 2n\pi - 2}{n^3\pi} = -\frac{4\pi}{n}$$

$$g(t) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nt}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

If $f(t)$ is the function of Problem 18, then for $0 < t < 2\pi$ we have

$$\frac{3t^2 - 6\pi t + 2\pi^2}{12} = \frac{1}{4}g(t) - \frac{\pi}{2}f(t) + \frac{\pi^2}{6}$$

$$= \frac{1}{4}\left(\frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nt}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nt}{n}\right) - \frac{\pi}{2}\left(\pi - 2\sum_{n=1}^{\infty}\frac{\sin nt}{n}\right) + \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{\cos nt}{n^2}.$$



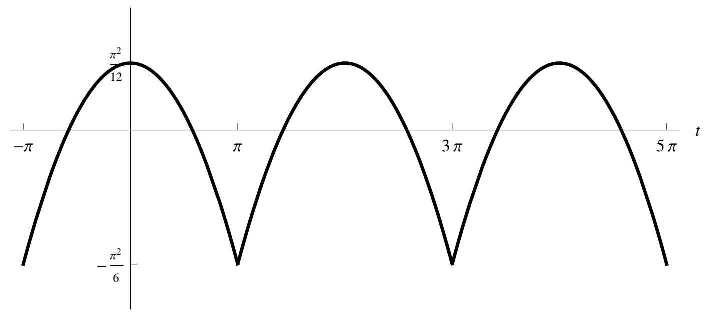
21. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{\pi^2}{3},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\,dt = \frac{4n\pi\cos n\pi + 2(n^2\pi^2-1)\sin n\pi}{n^3\pi} = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin nt\,dt = 0$$

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nt}{n^2} \qquad (-\pi < t < \pi)$$

$$\frac{\pi^2 - 3t^2}{12} = \frac{\pi^2}{12} - \frac{1}{4}\left(\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nt}{n^2}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nt}{n^2}$$ (figure on next page)

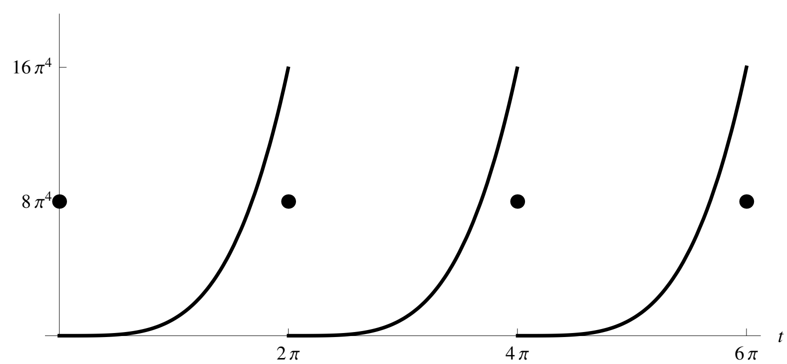


24. $$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^4\,dt = \frac{32\pi^4}{5}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^4 \cos nt\,dt$$
$$= \frac{8\left[2n\pi\left(2n^2\pi^2-3\right)\cos 2n\pi + (2n^4\pi^4-6n^2\pi^2+3)\sin 2n\pi\right]}{n^5\pi} = 16\left(\frac{2\pi^2}{n^2}-\frac{3}{n^4}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^4 \sin nt\,dt$$
$$= -\frac{8\left[(2n^4\pi^4-6n^2\pi^2+3)\cos 2n\pi + (6n\pi-4n^3\pi^3)\sin 2n\pi - 3\right]}{n^5\pi} = 16\pi\left(\frac{3}{n^2}-\frac{\pi^2}{n}\right)$$

$$f(t) = \frac{16\pi^4}{5} + 16\sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n^2}-\frac{3}{n^4}\right)\cos nt + 16\pi\sum_{n=1}^{\infty}\left(\frac{3}{n^2}-\frac{\pi^2}{n}\right)\sin nt$$ (figure below)



(b) When we substitute $t = 0, f(0) = 8\pi^4$ in the series of part (a) we get

$$8\pi^4 = \frac{16\pi^4}{5} + 32\pi^2\sum_{n=1}^{\infty}\frac{1}{n^2} - 48\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{16\pi^4}{5} + 32\pi^2\left(\frac{\pi^2}{6}\right) - 48\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

We now solve readily for $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$. Similarly, we find that $\sum_{n=1}^{\infty} (-1)^{n+1}/n^4 = 7\pi^4/720$ by substituting $t = \pi,\ f(\pi) = \pi^4$ in the series of part (a). Finally, addition of the first two series stated in part (b) yields the third one.

25. Now we want to sum the alternating series

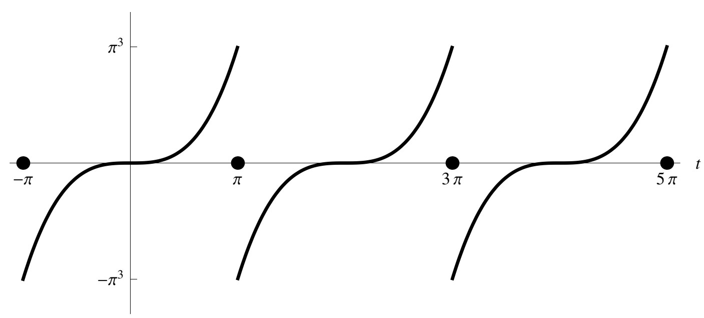
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \cdots$$

of reciprocals of odd cubes. Having used a Fourier series of t^4 in Problem 24 to evaluate $\Sigma(1/n^4)$, it is natural to look at a Fourier series of t^3. Let $f(t)$ be the period 2π function with $f(t) = t^3$ if $-\pi < t < \pi$. We calculate the Fourier coefficients of $f(t)$, and get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3\,dt = 0, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3 \cos nt\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3 \sin nt\,dt = -\frac{2n\pi\left(n^2\pi^2 - 6\right)\cos n\pi - 6(n^2\pi^2 - 2)\sin n\pi}{n^4\pi} = 2\left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)$$

$$t^3 = 2\pi^2 \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin nt}{n} - 12\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin nt}{n^3}. \qquad \text{(figure below)}$$



If we substitute $t = \pi/2$ and use Leibniz's series $\Sigma(-1)^{n+1}/n = \pi/4$ of Problem 17 we find that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \cdots = \frac{\pi^3}{32}.$$

There is *no* value of t whose substitution in the Fourier series of $f(t) = t^3$ yields the series $\Sigma(1/n^3)$ containing the reciprocal cubes of *both* the odd and even integers. Indeed, the summation in "closed form" of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots$$

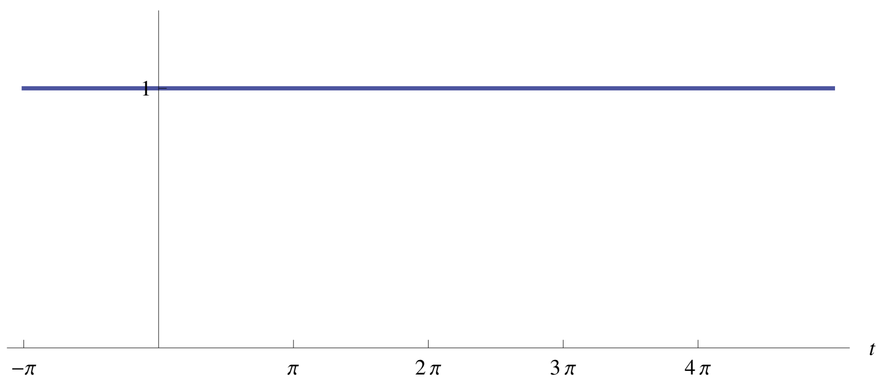
is a problem that has challenged many fine mathematicians since the time of Euler. Only in modern times (by R. Apery in 1978) has it been shown that this sum is an irrational number. For a delightful account of this work, see the article "A Proof that Euler Missed . . . An Informal Report" by Alfred van der Poorten in the *The Mathematical Intelligencer*, Volume 1 (1979), pages 195–203.

SECTION 9.3

FOURIER SINE AND COSINE SERIES

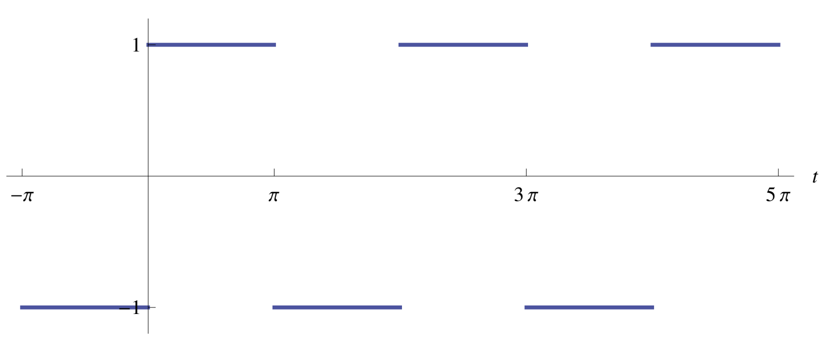
1. $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} 1\,dt = 2,\qquad a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \cos nt\,dt = \dfrac{2\sin n\pi}{n\pi} = 0$

Cosine series: $f(t) = 1$



$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nt\,dt = \frac{2(1-\cos n\pi)}{n\pi} = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

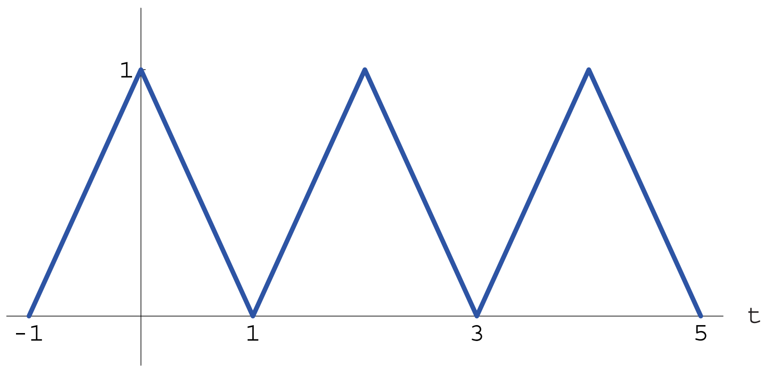
Sine series: $f(t) = \dfrac{4}{\pi}\left(\sin t + \dfrac{1}{3}\sin 3t + \dfrac{1}{5}\sin 5t + \dfrac{1}{7}\sin 7t + \cdots\right)$



2. $a_0 = 2\int_0^1 (1-t)\,dt = 1$

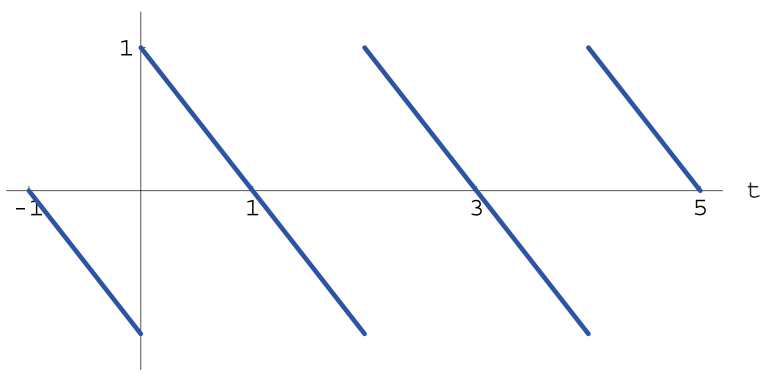
$a_n = 2\int_0^1 (1-t)\cos n\pi t\,dt = \dfrac{2(1-\cos n\pi)}{n^2\pi^2} = \dfrac{2}{n^2\pi^2}\left[1-(-1)^2\right]$

Cosine series: $f(t) = \dfrac{1}{2} + \dfrac{4}{\pi^2}\left(\cos \pi t + \dfrac{\cos 3\pi t}{3^2} + \dfrac{\cos 5\pi t}{5^2} + \dfrac{\cos 7\pi t}{7^2} + \cdots\right)$



$b_n = 2\int_0^1 (1-t)\sin n\pi t\,dt = \dfrac{2\left(n\pi - \sin n\pi\right)}{n^2\pi^2} = \dfrac{2}{n\pi}$

Sine series: $f(t) = \dfrac{2}{\pi}\left(\sin \pi t + \dfrac{\sin 2\pi t}{2} + \dfrac{\sin 3\pi t}{3} + \dfrac{\sin 4\pi t}{4} + \cdots\right)$

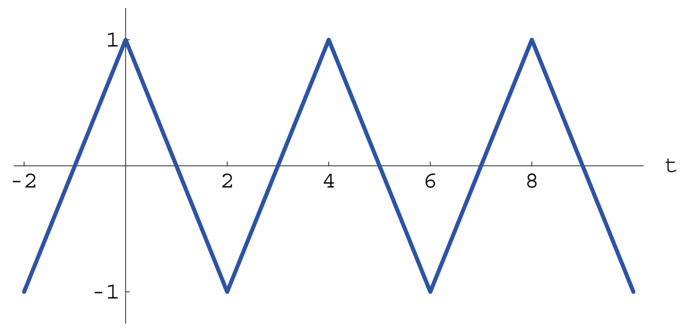


3. $a_0 = \int_0^2 (1-t)\,dt = 0$

$a_n = \int_0^2 (1-t)\cos\dfrac{n\pi t}{2}\,dt = \dfrac{4-4\cos n\pi - 2n\pi \sin n\pi}{n^2\pi^2} = \dfrac{4}{n^2\pi^2}\left[1-(-1)^2\right]$

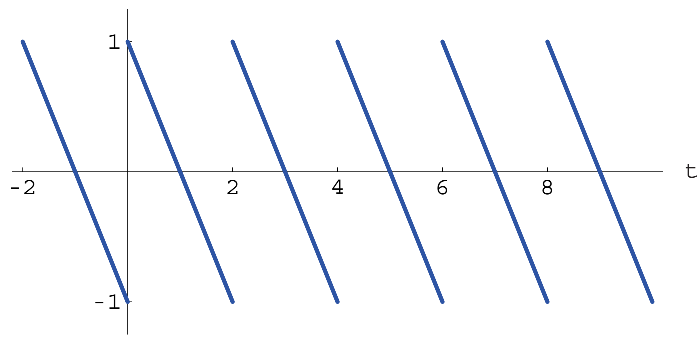
Cosine series: $f(t) = \dfrac{8}{\pi^2}\left(\cos\dfrac{\pi t}{2} + \dfrac{1}{3^2}\cos\dfrac{3\pi t}{2} + \dfrac{1}{5^2}\cos\dfrac{5\pi t}{2} + \dfrac{1}{7^2}\cos\dfrac{7\pi t}{2} + \cdots\right)$

See figure at top of next page.



$$b_n = \int_0^2 (1-t)\sin\frac{n\pi t}{2}\,dt = \frac{2n\pi(1+\cos n\pi) - 2\sin n\pi}{n^2\pi^2} = \frac{2}{n\pi}\left[1+(-1)^n\right]$$

Sine series: $f(t) = \frac{4}{\pi}\left(\frac{\sin\pi t}{2} + \frac{\sin 2\pi t}{4} + \frac{\sin 3\pi t}{6} + \frac{\sin 4\pi t}{8} + \cdots\right)$

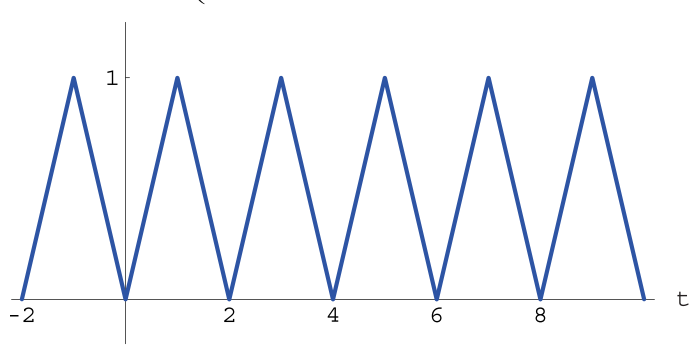


4. $a_0 = \int_0^1 t\,dt + \int_1^2 (2-t)\,dt = 1$

$$a_n = \int_0^1 t\cos\frac{n\pi t}{2}\,dt + \int_1^2 (2-t)\cos\frac{n\pi t}{2}\,dt$$

$$= \frac{16}{n^2\pi^2}\cos\frac{n\pi}{2}\sin^2\frac{n\pi}{4} = \begin{cases} 0 \text{ for } n \text{ odd} \\ 0 \text{ if } n = 4, 8, 12, \cdots \\ -16/n^2\pi^2 \text{ if } n = 2, 6, 10, \cdots \end{cases}$$

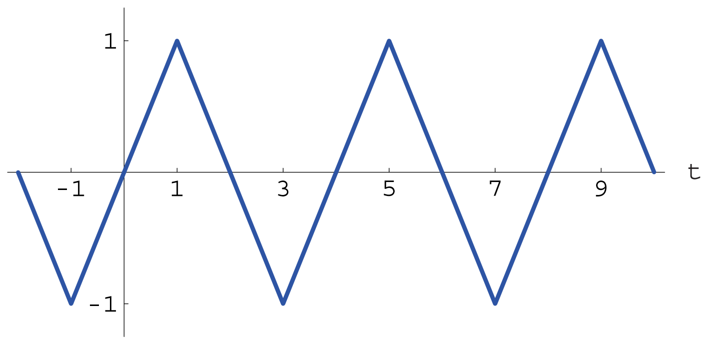
Cosine series: $f(t) = 1 - \frac{16}{\pi^2}\left(\frac{\cos\pi t}{2^2} + \frac{\cos 3\pi t}{6^2} + \frac{\cos 5\pi t}{10^2} + \frac{\cos 7\pi t}{14^2} + \cdots\right)$



$$b_n = \int_0^1 t\sin\frac{n\pi t}{2}\,dt + \int_1^2 (2-t)\sin\frac{n\pi t}{2}\,dt$$

$$= \frac{32}{n^2\pi^2}\cos\frac{n\pi}{4}\sin^3\frac{n\pi}{4} = \begin{cases} 0 \text{ for } n \text{ even} \\ +8/n^2\pi^2 \quad \text{if } n = 1, 5, 9, \cdots \\ -8/n^2\pi^2 \quad \text{if } n = 3, 7, 11, \cdots \end{cases}$$

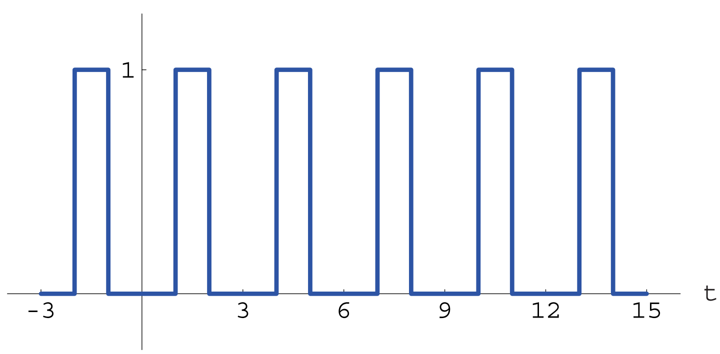
Sine series: $\quad f(t) = \dfrac{8}{\pi^2}\left(\sin\dfrac{\pi t}{2} - \dfrac{1}{3^2}\sin\dfrac{3\pi t}{2} + \dfrac{1}{5^2}\sin\dfrac{5\pi t}{2} - \dfrac{1}{7^2}\sin\dfrac{7\pi t}{2} + \cdots\right)$



5. $\quad a_0 = \dfrac{2}{3}\displaystyle\int_1^2 1\,dt = \dfrac{2}{3}$

$$a_n = \frac{2}{3}\int_1^2 \cos\frac{n\pi t}{3}\,dt = \frac{2}{n\pi}\left(\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3}\right) = \begin{cases} -2\sqrt{3}/n\pi \quad \text{if } n = 2, 8, 14, \cdots \\ +2\sqrt{3}/n\pi \quad \text{if } n = 4, 10, 16, \cdots \\ 0 \quad \text{otherwise} \end{cases}$$

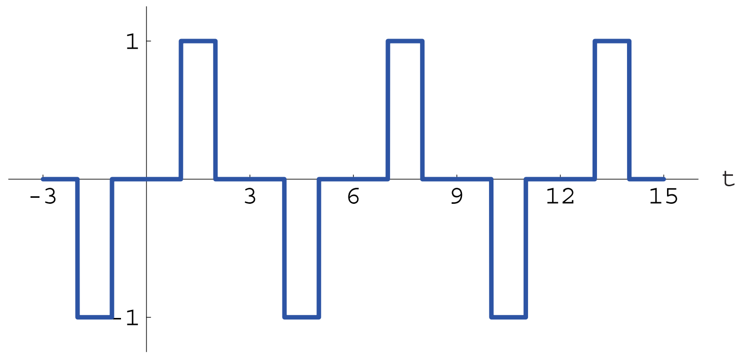
Cosine series: $\quad f(t) = \dfrac{1}{3} - \dfrac{2\sqrt{3}}{\pi}\left[\dfrac{1}{2}\cos\dfrac{2\pi t}{3} - \dfrac{1}{4}\cos\dfrac{4\pi t}{3} + \dfrac{1}{8}\cos\dfrac{8\pi t}{3} - \dfrac{1}{10}\cos\dfrac{10\pi t}{3} + \cdots\right]$



$$b_n = \frac{2}{3}\int_1^2 \sin\frac{n\pi t}{3}\,dt = \frac{2}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right) = \begin{cases} 0 \text{ for } n \text{ even} \\ +2/n\pi \quad \text{if } n = 1, 7, 13, \cdots \\ -4/n\pi \quad \text{if } n = 3, 9, 15, \cdots \\ +2/n\pi \quad \text{if } n = 5, 11, 17, \cdots \end{cases}$$

Sine series:

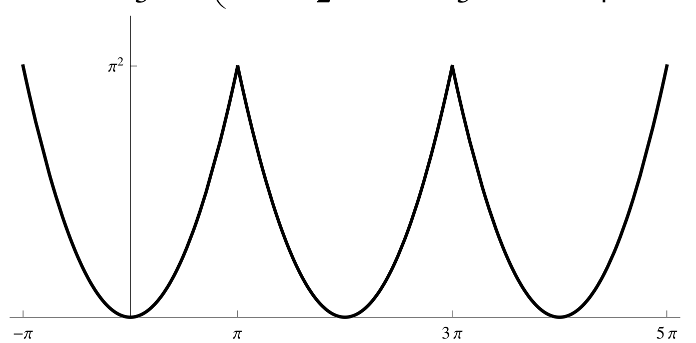
$$f(t) = \frac{2}{\pi}\left[\sin\frac{\pi t}{3} - \frac{2}{3}\sin\frac{3\pi t}{3} + \frac{1}{5}\sin\frac{5\pi t}{3} + \frac{1}{7}\sin\frac{7\pi t}{3} - \frac{2}{9}\sin\frac{9\pi t}{3} + \frac{1}{11}\sin\frac{11\pi t}{3} + \cdots\right]$$



6. $$a_0 = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \frac{2\pi^2}{3},$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t^2 \cos nt\,dt = \frac{2\left[2n\pi\cos n\pi + (n^2\pi^2 - 2)\sin n\pi\right]}{n^3\pi} = \frac{4(-1)^n}{n^2}$$

Cosine series: $$f(t) = \frac{\pi^2}{3} - 4\left(\cos t - \frac{1}{2^2}\cos 2t + \frac{1}{3^2}\cos 3t + \frac{1}{4^2}\cos 4t + \cdots\right)$$

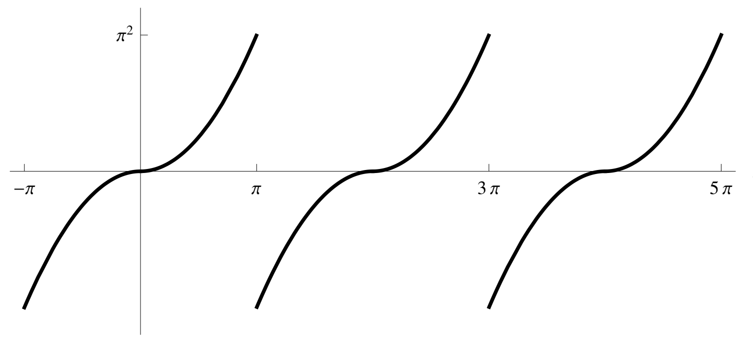


$$b_n = \frac{2}{\pi}\int_0^{\pi} t^2 \sin nt\,dt$$

$$= -\frac{2\left[(n^2\pi^2 - 2)\cos n\pi - 2n\pi\sin n\pi + 2\right]}{n^3\pi} = \begin{cases} +2\pi/n - 8/n^3\pi & \text{for } n \text{ odd} \\ -2\pi/n & \text{for } n \text{ even} \end{cases}$$

Sine series: $$f(t) = 2\pi\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \cdots\right)$$

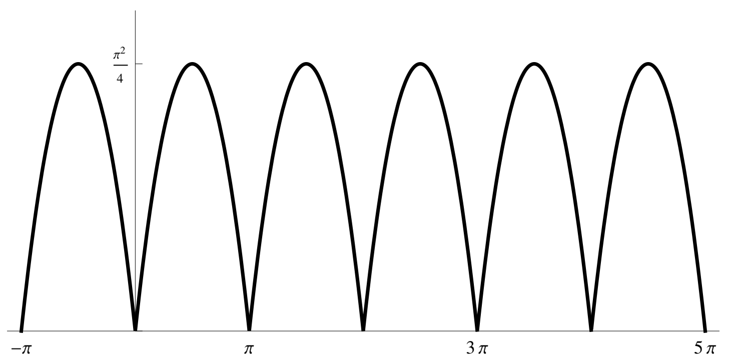
$$-\frac{8}{\pi}\left(\sin t + \frac{1}{3^3}\sin 3t + \frac{1}{5^3}\sin 5t + \frac{1}{7^3}\sin 7t + \cdots\right)$$



7. $$a_0 = \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\,dt = \frac{\pi^2}{3},$$

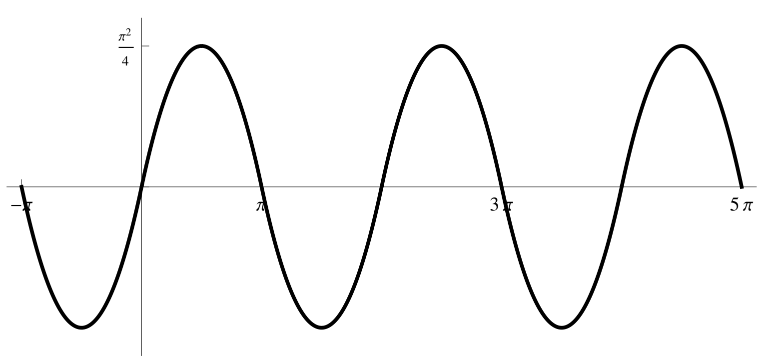
$$a_n = \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\cos nt\,dt = -\frac{2\left[n\pi\cos n\pi + n\pi - 2\sin n\pi\right]}{n^3\pi} = -\frac{2}{n^2}\left[1+(-1)^n\right]$$

Cosine series: $f(t) = \dfrac{\pi^2}{6} - 4\left(\dfrac{\cos 2t}{2^2} + \dfrac{\cos 4t}{4^2} + \dfrac{\cos 6t}{6^2} + \dfrac{\cos 8t}{8^2} + \cdots\right)$



$$b_n = \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\sin nt\,dt = \frac{2\left[2 - 2\cos n\pi - 2n\pi\sin n\pi\right]}{n^3\pi} = \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

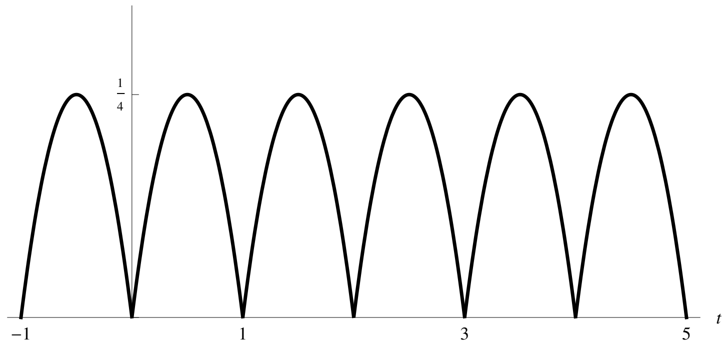
Sine series: $f(t) = \dfrac{8}{\pi}\left(\sin t + \dfrac{\sin 3t}{3^3} + \dfrac{\sin 5t}{5^3} + \dfrac{\sin 7t}{7^3} + \cdots\right)$



8. $a_0 = 2\int_0^1 (t-t^2)\,dt = \frac{1}{3},$

$$a_n = 2\int_0^1 (t-t^2)\cos n\pi t\,dt = -\frac{2[n\pi\cos n\pi + n\pi - 2\sin n\pi]}{n^3\pi^3} = -\frac{2}{n^2\pi^2}\left[1+(-1)^n\right]$$

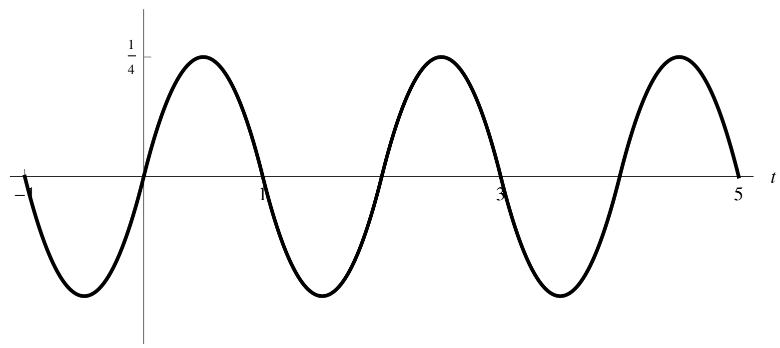
Cosine series: $f(t) = \frac{1}{6} - \frac{4}{\pi^2}\left(\frac{\cos 2\pi t}{2^2} + \frac{\cos 4\pi t}{4^2} + \frac{\cos 6\pi t}{6^2} + \frac{\cos 8\pi t}{8^2} + \cdots\right)$



$$b_n = 2\int_0^1 (t-t^2)\sin n\pi t\,dt = \frac{2[2-2\cos n\pi - n\pi\sin n\pi]}{n^3\pi^3} = \frac{4}{n^3\pi^3}\left[1-(-1)^n\right]$$

Sine series: $f(t) = \frac{8}{\pi^3}\left(\sin \pi t + \frac{\sin 3\pi t}{3^3} + \frac{\sin 5\pi t}{5^3} + \frac{\sin 7\pi t}{7^3} + \cdots\right)$

for $0 < t < 1$. Note that $t = 1/2$ yields the summation of Problem 25 in Section 9.2.

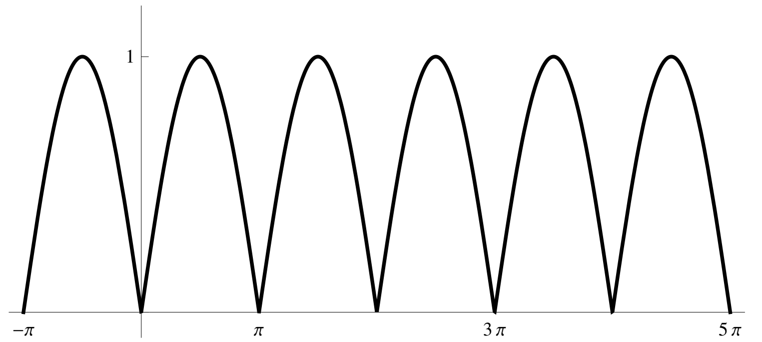


9. $a_0 = \frac{2}{\pi}\int_0^\pi \sin t\,dt = \frac{4}{\pi},$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin t\cos nt\,dt = \frac{2[1+\cos n\pi]}{\pi(1-n^2)} = \frac{2\left[1+(-1)^n\right]}{\pi(1-n^2)} \quad \text{if } n > 1$$

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin t\cos t\,dt = 0$$

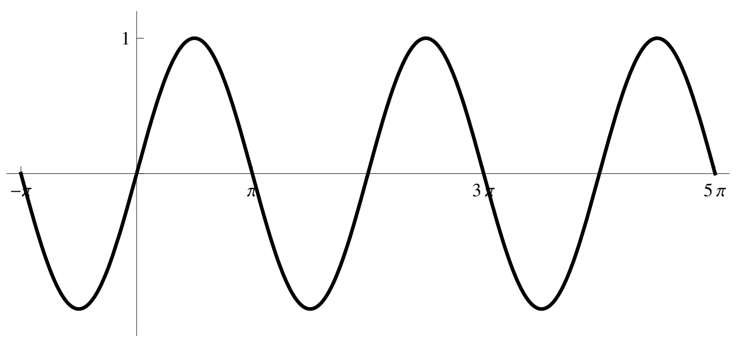
Cosine series: $f(t) = \frac{2}{\pi} - \frac{4}{\pi^2}\left(\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \frac{\cos 8t}{63} + \cdots\right)$



$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin t \sin nt\,dt = \frac{2\sin n\pi}{\pi(1-n^2)} = 0 \text{ if } n > 1$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 t\,dt = 1$$

Sine series: $f(t) = \sin t$



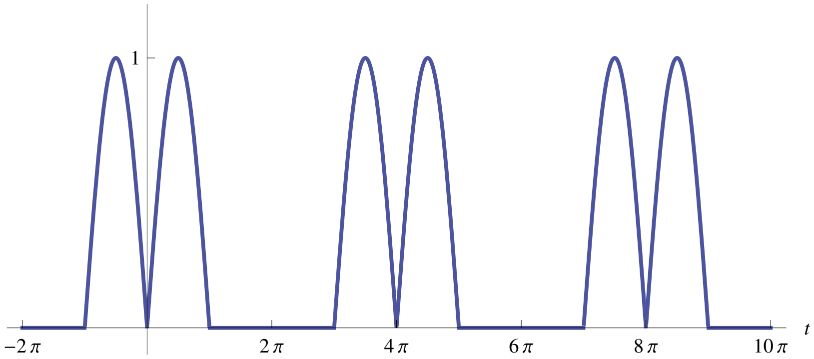
10. $$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin t\,dt = \frac{2}{\pi},$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin t \cos\frac{nt}{2}\,dt = -\frac{4\left[1+\cos\frac{n\pi}{2}\right]}{\pi(n^2-4)} = \begin{cases} -4/\pi(n^2-4) \text{ for } n \text{ odd} \\ -8/\pi(n^2-4) \text{ if } n = 4, 8, 12, \cdots \\ 0 \text{ if } n = 6, 10, 14, \cdots \end{cases}$$

$$a_2 = \frac{1}{\pi}\int_0^{\pi} \sin t \cos t\,dt = 0$$

Cosine series: $$f(t) = \frac{1}{\pi} - \frac{4}{\pi}\left(-\frac{1}{3}\cos\frac{t}{2} + \frac{1}{5}\cos\frac{3t}{2} + \frac{2}{12}\cos\frac{4t}{2} + \frac{1}{21}\cos\frac{5t}{2} + \frac{1}{45}\cos\frac{7t}{2} + \frac{2}{60}\cos\frac{8t}{2} + \cdots\right)$$

See the figure at the top of the next page.

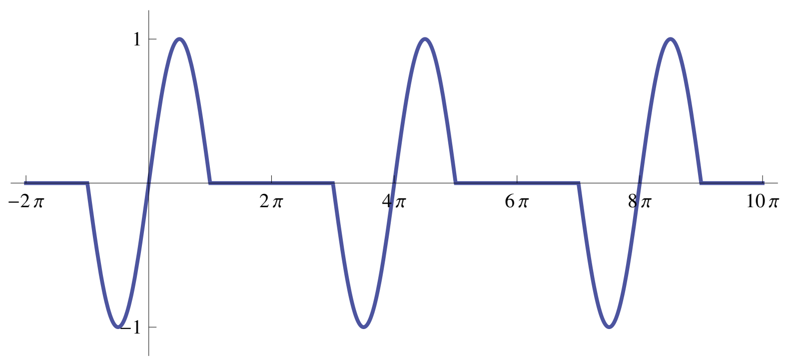


$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin t \sin\frac{nt}{2}\,dt = -\frac{4\sin\frac{n\pi}{2}}{\pi(n^2-4)} = \begin{cases} 0 \text{ for } n>2 \text{ even} \\ -4/\pi(n^2-4) \text{ if } n=1,5,9,\cdots \\ +4/\pi(n^2-4) \text{ if } n=3,7,11,\cdots \end{cases}$$

$$b_2 = \frac{1}{\pi}\int_0^{\pi} \sin^2 t\,dt = \frac{1}{2}$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 t\,dt = 1$$

Sine series: $f(t) = \frac{1}{2}\sin t - \frac{4}{\pi}\left(\frac{1}{3}\sin\frac{t}{2} + \frac{1}{5}\sin\frac{3t}{2} - \frac{1}{21}\sin\frac{5t}{2} + \frac{1}{45}\sin\frac{7t}{2} - \frac{1}{77}\sin\frac{9t}{2} + \cdots\right)$



11. In order to satisfy the endpoint conditions $x(0) = x(\pi) = 0$ we substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin nt$ and $1 = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}$ (from Example 1 in Section 9.1) into the differential equation $x'' + 2x = 1$. This gives

$$-\sum_{n=1}^{\infty} n^2 b_n \sin nt + 2\sum_{n=1}^{\infty} b_n \sin nt = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}.$$

We therefore choose $b_n = 4/\pi n(2-n^2)$ for n odd, $b_n = 0$ for n even. This gives the formal series solution

$$x(t) = \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin nt}{n(2-n^2)} = \frac{4}{\pi}\left(\sin t - \frac{\sin 3t}{21} - \frac{\sin 5t}{115} - \frac{\sin 7t}{329} - \cdots\right).$$

12. In order to satisfy the endpoint conditions $x(0) = x(\pi) = 0$ we substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin nt$ and $1 = \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin nt}{n}$ (from Example 1 in Section 9.1) into the differential equation $x'' - 4x = 1$. This gives

$$-\sum_{n=1}^{\infty} n^2 b_n \sin nt - 4\sum_{n=1}^{\infty} b_n \sin nt = \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin nt}{n}.$$

We therefore choose $b_n = -4/\pi n(n^2+4)$ for n odd, $b_n = 0$ for n even. This gives the formal series solution

$$x(t) = -\frac{4}{\pi}\left[\frac{\sin t}{5} + \frac{\sin 3t}{39} + \frac{\sin 5t}{145} + \frac{\sin 7t}{371} + \cdots\right].$$

13. In order to satisfy the endpoint conditions $x(0) = x(1) = 0$ we substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$ and $t = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin n\pi t}{n}$ (from Example 1 in Section 9.3, with $L = 1$) into the differential equation $x'' + x = t$. This gives

$$-\sum_{n=1}^{\infty} n^2\pi^2 b_n \sin n\pi t + \sum_{n=1}^{\infty} b_n \sin n\pi t = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin n\pi t}{n}.$$

We therefore choose $b_n = 2(-1)^{n+1}/\pi n(1-n^2\pi^2)$. This gives the formal series solution $x(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi t}{n(n^2\pi^2-1)}$ of our endpoint value problem.

14. In order to satisfy the endpoint conditions $x(0) = x(2) = 0$ we substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{2}$ and $t = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{2}$ (from Example 1 in Section 9.3 ,with $L = 2$) into the differential equation $x'' + 2x = t$. This gives

$$-\sum_{n=1}^{\infty} \frac{n^2\pi^2}{4} b_n \sin n\pi t + 2\sum_{n=1}^{\infty} b_n \sin n\pi t = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{2}.$$

We therefore choose

$$b_n = \frac{4(-1)^{n+1}/\pi n}{2 - n^2\pi^2/4} = \frac{16(-1)^{n+1}}{\pi n(8-n^2\pi^2)}$$

for n odd, $b_n = 0$ for n even. This gives the formal series solution

$$x(t) = \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin(n\pi t/2)}{n(n^2\pi^2 - 8)}$$ of our endpoint value problem.

15. In order to satisfy the endpoint conditions $x'(0) = x'(2) = 0$ we substitute the cosine series $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$ and $t = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos nt}{n^2}$ (from Example 1 in Section 9.3, with $L = \pi$) into the differential equation $x'' + 2x = t$. This gives

$$-\sum_{n=1}^{\infty} n^2 a_n \cos nt + a_0 + 2\sum_{n=1}^{\infty} a_n \cos nt = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos nt}{n^2}.$$

We therefore choose $a_0 = \pi/2$, $a_n = 0$ for $n > 0$ even, and $a_n = 4/\pi n^2(n^2 - 2)$ for n odd. This gives the formal series solution

$$x(t) = \frac{\pi}{4} + \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos nt}{n^2(n^2 - 2)} = \frac{\pi}{4} + \frac{4}{\pi}\left(-\cos t + \frac{\cos 3t}{63} + \frac{\cos 5t}{575} + \frac{\cos 7t}{2303} + \cdots\right)$$

of our endpoint value problem.

16. **(a)** Obviously $x_p(t) = t$ is a particular solution of $x'' + 4x = 4t$, so a general solution is given by $x(t) = A\cos 2t + B\sin 2t + t$. We satisfy the endpoint conditions $x(0) = x(2) = 0$ by choosing $A = 0$ and $B = -1/\sin 2$.

(b) The point is simply that the series in (31) is the Fourier sine series of the period 2 function defined by $f(t) = t - (\sin 2t)/(\sin 2)$ for $0 < t < 1$:

$$2\int_0^1 t \sin n\pi t \, dt = \frac{2[\sin n\pi - n\pi \cos n\pi]}{n^2\pi^2} = -\frac{2(-1)^n}{n\pi}$$

$$2\int_0^1 \sin 2t \sin n\pi t \, dt = \frac{2[2(\cos 2)\sin n\pi - n\pi(\sin 2)\cos n\pi]}{n^2\pi^2 - 4} = -\frac{2n\pi(-1)^n(\sin 2)}{n^2\pi^2 - 4}$$

$$b_n = -\frac{2(-1)^n}{n\pi} + \frac{2n\pi(-1)^n}{n^2\pi^2 - 4} = \frac{8(-1)^n}{n\pi(n^2\pi^2 - 4)}$$

$$t - \frac{\sin 2t}{\sin 2} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi t}{n(n^2\pi^2 - 4)} \quad \text{for} \quad 0 < t < 1$$

17. *Suggestion*: Substitute $u = -t$ in the left-hand integral.

18. The termwise derivative of the given Fourier series is

$$-(4/\pi)\ \Sigma\ (\sin n\pi t)/n - 4\ \Sigma\ \cos n\pi t.$$

But the series $\Sigma \cos n\pi t$ diverges at $t = 0$ (for instance). Hence the derived series does not converge to any function at all, let alone to $f'(t)$.

19. The first termwise integration yields

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n \cos nt}{n^2} + C_1,$$

and substitution of $t = 0$ gives $C_1 = 2\sum_{n=1}^{\infty}(-1)^{n+1}/n^2 = \pi^2/6,$ so

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n \cos nt}{n^2} + \frac{\pi^2}{6}.$$

A second termwise integration gives

$$\frac{t^3}{6} = 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin nt}{n^3} + \frac{\pi^2 t}{6} + C_2,$$

and substitution of $t = 0$ gives $C_2 = 0$. The final termwise integration gives

$$\frac{t^4}{24} = -2\sum_{n=1}^{\infty}\frac{(-1)^n \cos nt}{n^4} + \frac{\pi^2 t^2}{12} + C_3,$$

and substitution of $t = 0$ yields $C_3 = 2\sum_{n=1}^{\infty}(-1)^n/n^4.$

20. Substitution of $t = \pi$ in the formula of Problem 19 above gives

$$\frac{\pi^4}{24} = -2\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{\pi^4}{12} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4},$$

$$\frac{\pi^4}{24} = 2\sum_{n=1}^{\infty}\frac{1}{n^4} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = 4\sum_{n\text{ odd}}\frac{1}{n^4}$$

which gives $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$ Then

$$S = \sum_{n=1}^{\infty}\frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \cdots$$
$$= \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \cdots\right)$$

$$= \left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right)+\frac{1}{2^4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)$$

$$S = \left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right)+\frac{1}{15}S.$$

Solution of this last equation for S now gives

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{16}{15}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right) = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90}.$$

21. We want to calculate the coefficients in the period $4L$ Fourier sine series

$$F(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{2L}$$

which agrees with $f(t)$ if $0 < t < L$. Then

$$b_n = \frac{2}{2L}\int_0^L f(t)\sin\frac{n\pi t}{2L}\,dt + \frac{2}{2L}\int_L^{2L} f(2L-t)\sin\frac{n\pi t}{2L}\,dt.$$

The substitution $u = 2L - t$ yields

$$b_n = \frac{1}{L}\int_0^L f(t)\sin\frac{n\pi t}{2L}\,dt - \frac{1}{L}\int_L^0 f(u)\sin\frac{n\pi(2L-u)}{2L}\,du$$

$$= \frac{1}{L}\int_0^L f(t)\sin\frac{n\pi t}{2L}\,dt - \frac{(-1)^n}{L}\int_0^L f(u)\sin\frac{n\pi u}{2L}\,du.$$

Now it is clear that

$$b_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{2L}\,dt$$

if n is odd, whereas $b_n = 0$ if n is even.

22. We want to calculate the coefficients in the period $4L$ Fourier cosine series

$$G(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2L}$$

which agrees with $f(t)$ if $0 < t < L$. Then

$$a_n = \frac{2}{2L}\int_0^L f(t)\cos\frac{n\pi t}{2L}\,dt + \frac{2}{2L}\int_L^{2L} f(2L-t)\cos\frac{n\pi t}{2L}\,dt.$$

The substitution $u = 2L - t$ yields

$$\begin{aligned} a_n &= \frac{1}{L}\int_0^L f(t)\cos\frac{n\pi t}{2L}\,dt - \frac{1}{L}\int_L^0 f(u)\cos\frac{n\pi(2L-u)}{2L}\,du \\ &= \frac{1}{L}\int_0^L f(t)\cos\frac{n\pi t}{2L}\,dt - \frac{(-1)^n}{L}\int_0^L f(u)\cos\frac{n\pi u}{2L}\,du. \end{aligned}$$

Now it is clear that

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{2L}\,dt$$

if n is odd, whereas $a_n = 0$ if n is even (including $n = 0$).

23. $$b_n = \frac{2}{\pi}\int_0^\pi t\sin\frac{nt}{2}\,dt = \frac{4}{\pi n^2}\left(2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right) = \frac{8(-1)^{(n-1)/2}}{\pi n^2} \quad \text{for } n \text{ odd}$$

$$f(t) = \frac{8}{\pi^2}\left(\sin\frac{t}{2} - \frac{1}{3^2}\sin\frac{3t}{2} + \frac{1}{5^2}\sin\frac{5t}{2} - \frac{1}{7^2}\sin\frac{7t}{2} + \cdots\right)$$

24. In order to satisfy the endpoint conditions $x(0) = x'(\pi) = 0$ we substitute the odd half-multiple sine series $x(t) = \sum_{n\text{ odd}} b_n \sin\frac{nt}{2}$ and $t = \frac{8}{\pi}\sum_{n\text{ odd}}\frac{(-1)^{(n-1)/2}}{n^2}\sin\frac{nt}{2}$ (from Problem 21) into the differential equation $x'' - x = t$. This gives

$$-\sum_{n\text{ odd}}\frac{n^2 b_n}{4}\sin\frac{nt}{2} + \sum_{n\text{ odd}} b_n\sin\frac{nt}{2} = \frac{8}{\pi}\sum_{n\text{ odd}}\frac{(-1)^{(n-1)/2}}{n^2}\sin\frac{nt}{2}.$$

We therefore choose

$$b_n = \frac{8(-1)^{(n-1)/2}/\pi n^2}{1 - n^2/4} = \frac{32(-1)^{(n+1)/2}}{\pi n^2(n^2-4)}$$

for n odd. This gives the formal series solution

$$\begin{aligned} x(t) &= \frac{32}{\pi}\sum_{n\text{ odd}}\frac{(-1)^{(n+1)/2}}{n^2(n^2-4)}\sin\frac{nt}{2} \\ &= \frac{32}{\pi}\left(\frac{1}{3}\sin\frac{nt}{2} + \frac{1}{45}\sin\frac{3nt}{2} - \frac{1}{525}\sin\frac{5nt}{2} + \frac{1}{2205}\sin\frac{7nt}{2} - \cdots\right) \end{aligned}$$

of our endpoint value problem.

SECTION 9.4

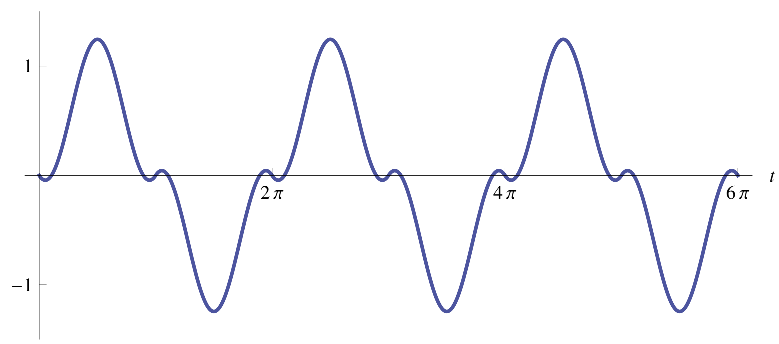
APPLICATIONS OF FOURIER SERIES

1. We substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin nt$ and $F(t) = \frac{12}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}$ (from Example 1 in Section 9.1) into the differential equation $x'' + 5x = F(t)$. This gives

$$-\sum_{n=1}^{\infty} n^2 b_n \sin nt + 5\sum_{n=1}^{\infty} b_n \sin nt = \frac{12}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}.$$

We therefore choose $b_n = 0$ for $n > 0$ even, and $b_n = 12/\pi n(5-n^2)$ for n odd. This gives the formal series solution

$$x_{sp}(t) = \frac{12}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n(5-n^2)} = \frac{12}{\pi}\left(\frac{\sin t}{4} - \frac{\sin 3t}{12} - \frac{\sin 5t}{100} - \frac{\sin 7t}{308} - \cdots\right).$$



2. We substitute the cosine series $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2}$ and

$F(t) = \frac{12}{\pi}\sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n}\cos\frac{n\pi t}{2}$ into the differential equation $x'' + 10x = F(t)$. This

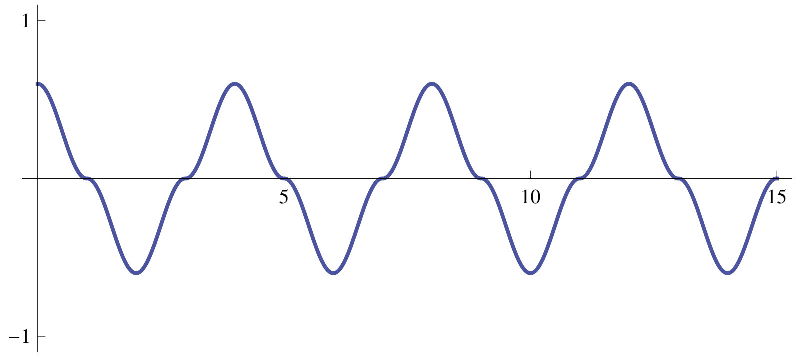
gives

$$-\sum_{n=1}^{\infty} \frac{n^2\pi^2}{4} a_n \cos\frac{n\pi t}{2} + 5a_0 + 10\sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2} = \frac{12}{\pi}\sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n}\cos\frac{n\pi t}{2}.$$

We therefore choose $a_0 = 0$ and $a_n = 0$ for $n > 0$ even, and

$$a_n = \frac{12(-1)^{(n-1)/2}/\pi n}{10 - \pi^2 n^2/4} = \frac{48(-1)^{(n-1)/2}}{\pi n(40 - \pi^2 n^2)}$$

for n odd. This gives the formal series solution $x_{sp}(t) = \dfrac{48}{\pi}\sum_{n \text{ odd}} \dfrac{(-1)^{(n-1)/2}}{n(40-\pi^2 n^2)} \cos\dfrac{n\pi t}{2}$.

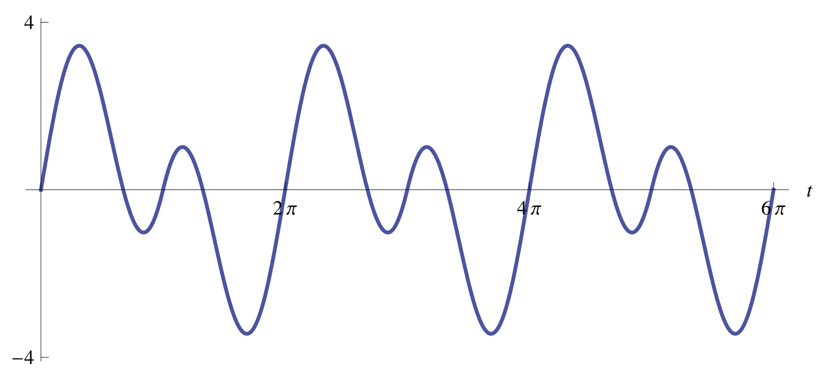


3. We substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin nt$ and $F(t) = 2\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}\sin nt}{n}$ (from Example 1 in Section 9.3, with $L = \pi$) into the differential equation $x'' + 3x = F(t)$. This gives

$$-\sum_{n=1}^{\infty} n^2 b_n \sin nt + 3\sum_{n=1}^{\infty} b_n \sin nt \;=\; 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nt}{n}.$$

We therefore choose $b_n = 4(-1)^{n-1}/n(3-n^2)$. This gives the formal series solution

$$x_{sp}(t) \;=\; 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nt}{n(3-n^2)} \;=\; 4\left(\frac{\sin t}{2} + \frac{\sin 2t}{2} - \frac{\sin 3t}{18} + \frac{\sin 4t}{52} - \cdots\right).$$



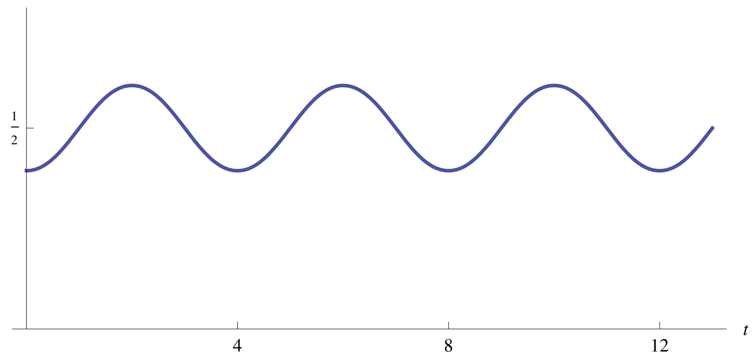
4. We substitute the cosine series $x(t) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi t}{2}$ and $F(t) = 2 - \dfrac{16}{\pi^2}\sum_{n \text{ odd}} \dfrac{1}{n^2}\cos\dfrac{n\pi t}{2}$ (from Example 1 in Section 9.3, with $L = 2$) into the differential equation $x'' + 4x = F(t)$. This gives

$$-\sum_{n=1}^{\infty}\frac{n^2\pi^2}{4}a_n\cos\frac{n\pi t}{2}+2a_0+4\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{2} = 2-\frac{16}{\pi^2}\sum_{n\text{ odd}}\frac{1}{n^2}\cos\frac{n\pi t}{2}.$$

We therefore choose $a_0 = 1$ and $a_n = 0$ for $n > 0$ even, and

$$a_n = \frac{-16/\pi^2 n^2}{4-\pi^2 n^2/4} = -\frac{64}{\pi^2 n^2(16-\pi^2 n^2)}$$

for n odd. This gives the formal series solution $x_{\text{sp}}(t) = \dfrac{1}{2}-\dfrac{64}{\pi^2}\sum_{n\text{ odd}}\dfrac{\cos n\pi t/2}{\pi^2 n^2(16-\pi^2 n^2)}$.

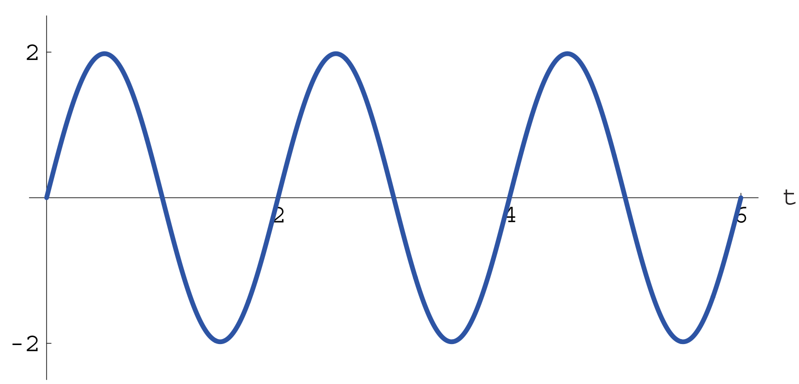


5. We substitute the sine series $x(t)=\sum_{n=1}^{\infty}b_n\sin n\pi t$ and $F(t)=\dfrac{8}{\pi^3}\sum_{n\text{ odd}}\dfrac{\sin n\pi t}{n^3}$ into the differential equation $x''+10x=F(t)$. This gives

$$-\sum_{n=1}^{\infty}n^2\pi^2 b_n\sin n\pi t+10\sum_{n=1}^{\infty}b_n\sin n\pi t = \frac{8}{\pi^3}\sum_{n\text{ odd}}\frac{\sin n\pi t}{n^3}.$$

We therefore choose $b_n = 8/n^3\pi^3(10-n^2\pi^2)$. This gives the formal series solution

$$x_{\text{sp}}(t) = \frac{8}{\pi^3}\sum_{n\text{ odd}}\frac{\sin n\pi t}{n^3(10-n^2\pi^2)}.$$



6. We substitute the cosine series $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$ and $F(t) = \frac{4}{\pi} - \frac{4}{\pi}\sum_{n \text{ even}} \frac{\cos nt}{n^2 - 1}$ into the differential equation $x'' + 2x = F(t)$. This gives

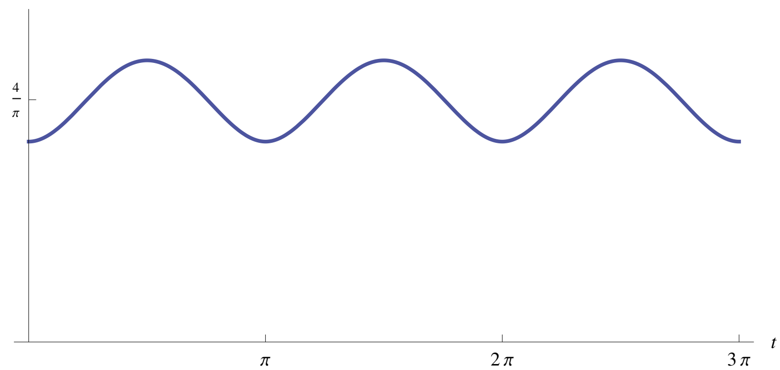
$$-\sum_{n=1}^{\infty} n^2 a_n \cos nt + a_0 + 2\sum_{n=1}^{\infty} a_n \cos nt = \frac{4}{\pi} - \frac{4}{\pi}\sum_{n \text{ even}} \frac{\cos nt}{n^2 - 1}.$$

We therefore choose $a_0 = 4/\pi$ and $a_n = 0$ for n odd, and

$$a_n = \frac{-4/\pi(n^2-1)}{2-n^2} = \frac{4}{\pi(n^2-1)(n^2-2)}$$

for n even. This gives the formal series solution

$$x_{sp}(t) = \frac{4}{\pi} - \frac{4}{\pi}\sum_{n \text{ even}} \frac{\cos nt}{(n^2-1)(n^2-2)} = \frac{4}{\pi}\left(1 - \frac{\cos 2t}{6} - \frac{\cos 4t}{210} - \frac{\cos 6t}{1190} - \cdots\right).$$



In Problems 7–12 we are dealing with the equation $mx'' + kx = F(t)$ where $F(t)$ is the external periodic force. The natural frequency is $\omega_0 = \sqrt{k/m}$. If the Fourier series of $F(t)$ contains a term of the form $\cos(N\pi t/L)$ or $\sin(N\pi t/L)$ with $\omega_0 = N\pi/L$, then pure resonance occurs. Otherwise, it does not.

7. The natural frequency is $\omega_0 = 3$, and

$$F(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \cdots\right).$$

Thus the Fourier series of $F(t)$ contains a $\sin 3t$ term, so resonance does occur.

8. The natural frequency is $\omega_0 = \sqrt{5}$, and $F(t) = \Sigma\, b_n \sin n\pi t$. Since $n\pi \neq \sqrt{5}$ for any integer n, pure resonance does not occur.

9. The natural frequency is $\omega_0 = 2$, and

$$F(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \cdots\right).$$

Because the $\sin 2t$ term is missing from the Fourier series of $F(t)$, resonance will not occur.

10. The natural frequency is $\omega_0 = 2\pi$. From Equation (16) in Section 9.3 of the text we see that the Fourier series of $F(t)$ contains a $\sin 2\pi t$ term. Hence pure resonance occurs.

11. The natural frequency is $\omega_0 = 4$. From Equation (15) in Section 9.3 we see that

$$F(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \cdots\right).$$

Because the $\cos 4t$ term is missing, we see that resonance will not occur.

12. The natural frequency is $\omega_0 = 5$, and the Fourier series of $F(t)$ is of the form $F(t) = \Sigma\, b_n \sin nt$. We calculate b_5, and find that

$$b_5 = \frac{2}{\pi}\int_0^{\pi} (\pi t - t^2)\sin 5t\, dt = \frac{4 - 4\cos 5\pi - 10\pi \sin 5\pi}{125\pi} = \frac{8}{125\pi} \neq 0,$$

Thus the term sin $5t$ is present in $F(t)$, and so pure resonance occurs.

Problems 13–18 are based on Equations (14)–(16) in the text, according to which the steady periodic solution of

$$mx'' + cx' + kx = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi t}{L}$$

is given by

$$x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin(\omega_n t - \alpha_n),$$

where

$$\omega_n = \frac{n\pi}{L},$$

$$\alpha_n = \tan^{-1}\frac{c\omega_n}{k - m\omega_n^2} \quad \text{in the interval } [0, \pi],$$

$$b_n = \frac{B_n}{\sqrt{\left(k - m\omega_n^2\right)^2 + \left(c\omega_n\right)^2}}.$$

This calculation is readily automated. The following MATLAB script was written to calculate the coefficients $\{b_n\}$ for Problem 13. Only the values of m, c, k, L and the calculation of the force function coefficients $\{B_n\}$ need to be changed for Problems 14–18.

```
m = 1;   c = 0.1;   k = 4;
L = pi;
results = ones(0,4);
for n = 1:9
   w = n*pi/L;
   alpha = atan(c*w/(k-m*w^2));
   if k-m*w^2<0
      alpha = pi + alpha;
   end
   B = 12/(pi*n);          % force function coeffs
   if floor(n/2)==n/2      % are nonzero if n is odd,
      B = 0;               % zero if n is even
   end
   b = B/sqrt((k-m*w^2)^2+(c*w)^2);
   results = [results; n, b, w, alpha];
end
results
```

13. $B_n = 12/\pi n$ for n odd, $B_n = 0$ for n even

$$x_{sp}(t) \approx 1.2725 \sin(t - 0.0333) + 0.2542 \sin(3t - 3.0817) + 0.0364 \sin(5t - 3.1178) + \cdots$$

14. $B_n = 4(-1)^{n+1}/n$ for $n = 1, 2, 3, \cdots$

$$x_{sp}(t) \approx 0.2500 \sin(t - 0.0063) - 0.2000 \sin(2t - 0.0200) + 4.444 \sin(3t - 1.5708) - 0.0714 \sin(4t - 3.1130) + \cdots$$

Note the dominance of the $n = 3$ term.

15. $B_n = 8/n^3\pi^3$ for n odd, $B_n = 0$ for n even

$$x_{sp}(t) \approx 0.08150 \sin(\pi t - 1.44692) + 0.00004 \sin(3\pi t - 3.10176) + \cdots$$

16. $F(t) = A_0 + \Sigma A_n \cos(n\pi t/2)$ where $A_0 = 2$, $A_n = -16/\pi^2 n^2$ for n odd, $A_n = 0$ for n even and positive.

$$x_{sp}(t) \approx 0.5000 + 1.0577 \cos(\pi t/2 - 0.0103) - 0.0099 \cos(3\pi t/2 - 3.1390) - 0.0011 \cos(5\pi t/2 - 3.1402) \cdots$$

17. $B_n = 60/n\pi$ for n odd, $B_n = 0$ for n even

$$x_{sp}(t) \approx 0.5687\sin(\pi t - 0.0562) + 0.4271\sin(3\pi t - 0.3891) + 0.1396\sin(5\pi t - 2.7899) + 0.0318\sin(7\pi t - 2.9874) + \cdots$$

$x_{sp}(5) \approx 0.248$ ft ≈ 2.98 in.

18. $B_n = (4/\pi n^2)\sin(n\pi/2)$

$$x_{sp}(t) \approx 0.0531\sin(t - 0.0004) - 0.0088\sin(3t - 0.0019) + 1.0186\sin(5t - 1.5708) - 0.0011\sin(7t - 3.1387) + \cdots$$

Note the dominance of the $n = 5$ term.

19. We suppose that $f(t+P) = f(t)$ and $g(t+Q) = g(t)$ for all t. If $P/Q = m/n$ where m and n are integers, let $R = nP = mQ$. Then

$$f(t+R) + g(t+R) = f(t+nP) + g(t+mQ) = f(t) + g(t)$$

for all t, so we see that the sum $f(t) + g(t)$ is periodic with period R.

20. Suppose that the function $f(t) = \cos pt + \cos qt$ is periodic and has period $L > 0$, so $f(t+L) = f(t)$ for all t. That is,

$$\cos p(t+L) + \cos q(t+L) = \cos pt + \cos qt .$$

Substitution of $t = 0$ then gives $\cos pL + \cos qL = 2\cos 0 = 2$, which implies that $\cos pL = \cos qL = 1$ (why?). It follows that pL and qL must both be nonzero integral multiples of 2π. Then

$$pL = 2m\pi \text{ and } qL = 2n\pi \quad \Rightarrow \quad \frac{pL}{qL} = \frac{2m\pi}{2n\pi} \quad \Rightarrow \quad \frac{p}{q} = \frac{m}{n}$$

with m and n integers. Thus p/q must be rational if the sum $f(t) = \cos pt + \cos qt$ is to be periodic.

SECTION 9.5

HEAT CONDUCTION AND SEPARATION OF VARIABLES

1. From Equation (31) in the text, with $L = \pi$ and $k = 3$, we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp(-3n^2 t)\sin nx\,.$$

With $b_2 = 4$ and $b_n = 0$ otherwise we get the solution

$$u(x,\,t) = 4e^{-12t}\sin 2x.$$

2. From Equation (40) in the text, with $k = 10$ and $L = 5$ we get,

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 t}{25}\right)\cos\frac{n\pi x}{5}.$$

With $a_0 = 14$ and $a_n = 0$ for $n > 0$ we get the solution $u(x,\,t) = 7$ (constant).

3. With $L = 1$ and $k = 2$ in Equation (31), we take $b_1 = 5$, $b_3 = -1/5$, and $b_n = 0$ otherwise. The result is the solution

$$u(x,t) = 5e^{-2\pi^2 t}\sin \pi x - \frac{1}{5}e^{-18\pi^2 t}\sin 3\pi x.$$

4. From Equation (31) with $k = 1$ and $L = \pi$ we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp(-n^2 t)\sin nx\,.$$

But the $\sin A \cos B$ identity yields

$$4 \sin 4x \cos 2x = 2 \sin 2x + 2 \sin 6x.$$

Hence we choose $b_2 = b_6 = 2$ and $b_n = 0$ for $n \neq 2, 6$. Thus

$$u(x,\,t) = 2e^{-4t}\sin 2x + 2e^{-36t}\sin 6x.$$

5. From Equation (40) in the text, with $k = 2$ and $L = 3$ we get

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{2n^2\pi^2 t}{9}\right)\cos\frac{n\pi x}{3}.$$

With $a_0 = 0$, $a_2 = 4$, $a_4 = -2$, and $a_n = 0$ otherwise, and $a_n = 0$ we get the solution

$$u(x,t) = 4\exp\left(-\frac{8\pi^2 t}{9}\right)\cos\frac{2\pi x}{3} - 2\exp\left(-\frac{32\pi^2 t}{9}\right)\cos\frac{4\pi x}{3}.$$

6. From Equation (31) with $k = 1/2$ and $L = 1$ we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 t}{2}\right)\sin n\pi x.$$

Trigonometric identities yield

$$\begin{aligned} 4 \sin \pi x \cos^3 \pi x &= (2 \sin \pi x \cos \pi x)(2 \cos^2 \pi x) \\ &= (\sin 2\pi x)(1 + \cos 2\pi x) = \sin 2\pi x + (1/2)\sin 4\pi x. \end{aligned}$$

Hence we choose $b_2 = 1$, $b_4 = 1/2$, and $b_n = 0$ otherwise to get

$$u(x, t) = \exp(-2\pi^2 t) \sin 2\pi x + (1/2)\exp(-8\pi^2 t) \sin 4\pi x.$$

7. From Equation (40) in the text, with $k = 1/3$ and $L = 2$ we get,

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 t}{12}\right)\cos\frac{n\pi x}{2}.$$

Because of the identity $\cos^2 2\pi x = (1 + \cos 4\pi x)/2$, we choose $a_0 = 1$, $a_8 = 1/2$, and $a_n = 0$ otherwise. This gives the solution

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\exp\left(-\frac{16\pi^2 t}{3}\right)\cos 4\pi x.$$

8. From Equation (40) with $k = 1$ and $L = 2$ we get

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 t}{4}\right)\cos\frac{n\pi x}{2}.$$

But

$$10 \cos \pi x \cos 3\pi x = 5 \cos 2\pi x + 5 \cos 4\pi x.$$

Hence we choose $b_4 = b_8 = 5$ and $b_n = 0$ otherwise to get the solution

$$u(x, t) = 5 \exp(-4\pi^2 t) \cos 2\pi x + 5 \exp(-16\pi^2 t) \cos 4\pi x.$$

9. Because of the zero endpoint conditions $u(0,t) = u(5,t) = 0$, we use the Fourier sine series expansion

$$u(x,0) = \frac{100}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\frac{n\pi x}{5}$$

of $u(x,0) = 25$ on the interval $0 < x < 5$. When we supply the exponential factors in Eq. (31) with $k = 1/10$ and $L = 5$, we get the solution

$$u(x,t) = \frac{100}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\exp\left(\frac{-n^2\pi^2 t}{250}\right)\sin\frac{n\pi x}{5}$$

10. Because of the zero endpoint conditions $u(0,t) = u(10,t) = 0,$ we use the Fourier sine series expansion

$$u(x,0) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{10}$$

of $u(x,0) = 4x$ on the interval $0 < x < 10$ (from Eq. (16) in Section 9.3). When we supply the exponential factors in Eq. (31) here with $k = 1/5$ and $L = 10,$ we get

$$u(x,t) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp\left(-\frac{n^2\pi^2 t}{500}\right)\sin\frac{n\pi x}{10}.$$

11. Because of the zero-derivative endpoint conditions $u_x(0,t) = u_x(10,t) = 0,$ we use the Fourier cosine series expansion

$$u(x,0) = 20 - \frac{160}{\pi^2}\sum_{n \text{ odd}}\frac{1}{n^2}\cos\frac{n\pi x}{10}$$

of $u(x,0) = 4x$ on the interval $0 < x < 10$ (from Eq. (15) in Section 9.3). When we supply the exponential factors in Eq. (40) here with $k = 1/5$ and $L = 10,$ we get

$$u(x,t) = 20 - \frac{160}{\pi^2}\sum_{n \text{ odd}}\frac{1}{n^2}\exp\left(\frac{-n^2\pi^2 t}{500}\right)\cos\frac{n\pi x}{10}$$

12. From Equation (31) with $k = 1$ and $L = 100$ we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 t}{10000}\right)\sin\frac{n\pi x}{100}.$$

Because of the zero endpoint conditions $u(0,t) = u(100,t) = 0,$ the $\{b_n\}$ should be the Fourier sine coefficients of $f(x) = x(100 - x)$ on $[0, 100]$, given by

$$\begin{aligned} b_n &= \frac{2}{100}\int_0^{100} x(100-x)\sin\frac{n\pi x}{100}\,dx \\ &= \frac{20000(2 - 2\cos n\pi - n\pi\sin n\pi)}{n^3\pi^3} = \begin{cases} 80000/n^3\pi^3 & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ odd}. \end{cases} \end{aligned}$$

This gives the solution

$$u(x,t) = \frac{80000}{\pi^3}\sum_{n \text{ odd}}\frac{1}{n^3}\exp\left(-\frac{n^2\pi^2 t}{10000}\right)\sin\frac{n\pi x}{100}.$$

13. **(a)** The boundary value problem is

$$u_t = ku_{xx} \quad (0 < x < 40),$$
$$u_x(0, t) = u_x(40, t) = 0,$$
$$u(x, 0) = 100.$$

By Equation (31) in the text (with $L = 40$) the solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 kt}{1600}\right)\sin\frac{n\pi x}{40}.$$

We use the Fourier sine coefficients $b_n = 400/\pi n$ for n odd, $b_n = 0$ otherwise, of the initial value function $f(x) = 100$ on the interval $0 < x < 100$. This gives

$$u(x,t) = \frac{400}{\pi}\sum_{n\text{ odd}} \frac{1}{n}\exp\left(\frac{-n^2\pi^2 kt}{1600}\right)\sin\frac{n\pi x}{40}.$$

(b) With $k = 1.15$ for copper we find that

$$u(20, 300) \approx 15.1591 - 0.000000204 + \cdots \approx 15.16°C.$$

(c) With $k = 0.005$ for concrete, the first term of the series gives

$$u(20,t) = \frac{400}{\pi}\exp\left(-\frac{0.00\pi^2 t}{1600}\right) = 15,$$

and we solve for $t \approx 66{,}342 \text{ sec} \approx 19 \text{ hr } 15 \text{ min } 42 \text{ sec}$. As a check that the first term suffices for this computation, we find that the next term in the series is then approximately 0.00000019.

14. **(a)** The boundary value problem is

$$u_t = ku_{xx} \quad (0 < x < 50)$$
$$u_x(0, t) = u_x(50, t) = 0$$
$$u(x, 0) = 2x$$

with $k = 1.15$ cm^2/sec for copper. By Equation (40) in the text (with $L = 50$) the solution is of the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 kt}{2500}\right)\cos\frac{n\pi x}{50}.$$

Consulting the Fourier series given in Equation (15) of Section 9.2, we satisfy the initial condition $u(x, 0) = 2x$ by choosing $a_0 = 100$, $a_n = -400/n^2\pi^2$ for n odd, and

$a_n = 0$ for n even. Thus

$$u(x,t) = 50 - \frac{400}{\pi^2}\sum_{n\text{ odd}}\frac{1}{n^2}\exp\left(-\frac{n^2\pi^2kt}{2500}\right)\cos\frac{n\pi x}{50}.$$

(b) With $k = 1.15$ for copper we find that

$$u(10,60) \approx 50 - 24.9698 + 0.1199 + 0.0018 + 0.0000 - \cdots \approx 25.15°C.$$

(c) To find out how long it takes the temperature to reach 45°C at the point $x = 10$, we solve the equation

$$50 - \frac{400}{\pi^2}\exp\left(-\frac{1.15\pi^2 t}{2500}\right)\cos\frac{\pi}{5} = 45$$

that we get upon retaining only the first two terms of the series above (with $x = 10$). Using logarithms (for instance) we find that $t \approx 414.23\text{ sec} \approx 6\text{ min }54\text{ sec}$. To confirm that two terms suffice for this calculation, we retain 3 terms and use a computer or calculator to solve the equation

$$50 - \frac{400}{\pi^2}\exp\left(-\frac{1.15\pi^2 t}{2500}\right)\cos\frac{\pi}{5} - \frac{400}{9\pi^2}\exp\left(-\frac{9\times 1.15\pi^2 t}{2500}\right)\cos\frac{3\pi}{5} = 45$$

for (again) $t \approx 414.23\text{ sec}$.

15. We need only calculate the coefficients in the usual zero-endpoint series

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2kt}{L^2}\right)\sin\frac{n\pi x}{L}.$$

For the function $f(x) \equiv A$ for $0 < x < L/2$, $f(x) \equiv 0$ for $L/2 < x < L$ we calculate the Fourier sine coefficient

$$b_n = \frac{2}{L}\int_0^{L/2} A\sin\frac{n\pi x}{L}\,dx = \frac{4A}{n\pi}\sin^2\frac{n\pi}{4} = \frac{4A}{n\pi}\times\begin{cases}1/2 & \text{for } n \text{ odd},\\ 1 & \text{for } n = 2, 6, 10, \cdots,\\ 0 & \text{for } n = 4, 8, 12, \cdots.\end{cases}$$

16. **(a)** Summing numerically the series in Problem 15 with the values $k = 0.15$ for iron, $L = 50$, $A = 100$, $x = 25$, and $t = 1800$, we find that

$$u(25,1800) \approx 21.9259 - 0.0014 + 0.0000 - \cdots \approx 21.9245 \approx 22°\text{C}.$$

(b) Because for x fixed the temperature is a function of the ***product*** kt, in the case of concrete slabs with $k = 0.005$ the same temperature will be attained when

$$(0.005)(t) = (0.15)(1800),$$

that is, when $t = 54000$ sec $= 15$ hr.

SECTION 9.6

VIBRATING STRINGS AND THE ONE-DIMENSIONAL WAVE EQUATION

In Problems 1–10 we use the general solution

$$y(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi at}{L} + B_n \sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L} \qquad (*)$$

of the string equation $y_{tt} = a^2 y_{xx}$ with endpoint conditions $y(0, t) = y(L, t) = 0$. This form of the solution is obtained by superposition of the solutions in Equations (23) and (33) of Problems A and B in this section. It remains only to choose the coefficients $\{A_n\}$ and $\{B_n\}$ so as to satisfy given initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x), \quad \text{thus,} \quad A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx; \text{ and}$$

$$y_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} B_n \sin\frac{n\pi x}{L} = g(x), \quad \text{thus,} \quad B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

1. Here $a = 2$ and $L = \pi$. To satisfy the condition $y(x, 0) = (1/10)\sin 2x$ we choose $A_2 = 1/10$ in Eq. (*) above, and $A_n = 0$ otherwise. To satisfy the condition $y_t(x, 0) = 0$ we choose $B_n = 0$ for all n. Thus

$$y(x, t) = \frac{1}{10}\cos 4t \sin 2x.$$

2. Here $a = L = 1$. To satisfy the condition

$$y(x, 0) = \frac{1}{10}\sin \pi x - \frac{1}{20}\sin 3\pi x$$

we choose $A_1 = 1/10$ and $A_3 = -1/20$ in Eq. (*) above, and $A_n = 0$ otherwise. To satisfy the condition $y_t(x, 0) = 0$ we choose $B_n = 0$ for all n. Thus

$$y(x,\, t) \;=\; \frac{1}{10}\cos \pi t \sin \pi x - \frac{1}{20}\cos 3\pi t \sin 3\pi x.$$

3. Here $a = 1/2$ and $L = \pi.$ Choosing $A_1 = 1/10$ and $A_n = 0$ otherwise, $B_1 = 1/5$ and $B_n = 0$ otherwise, we get

$$y(x,\, t) \;=\; \frac{1}{10}\left(\cos\frac{t}{2} + 2\sin\frac{t}{2}\right)\sin x.$$

4. Here $a = 1/2$ and $L = 2,$ so $n\pi x/L = n\pi x/2$ and $n\pi at/L = n\pi t/4.$ To satisfy the condition

$$y(x,\, 0) \;=\; \frac{1}{5}\sin \pi x \cos \pi x \;=\; \frac{1}{10}\sin 2\pi x \;=\; \frac{1}{10}\sin\frac{4\pi x}{2},$$

we choose $A_4 = 1/10$ and $A_n = 0$ for $n \neq 4.$ To satisfy the condition $y_t(x,\, 0) = 0$ we choose $B_n = 0$ for all $n.$ Thus

$$y(x,\, t) \;=\; \frac{1}{10}\cos \pi t \sin 2\pi x.$$

5. Here $a = 5$ and $L = 3.$ Choosing $A_3 = 1/4$ and $A_n = 0$ for $n \neq 3,$ $B_6 = 1/\pi$ and $B_n = 0$ for $n \neq 6,$ we get

$$y(x,\, t) \;=\; \frac{1}{4}\cos 5\pi t \sin \pi x + \frac{1}{\pi}\sin 10\pi t \sin 2\pi x.$$

6. Here $a = 10$ and $L = \pi.$ To satisfy the condition $y_t(x,\, 0) = 0$ we choose $B_n = 0$ for all $n,$ so

$$y(x,\, t) \;=\; \sum_{n=1}^{\infty} A_n \cos 10nt \sin nx.$$

To satisfy the condition $y(x,\, 0) = x(\pi - x)$ we choose

$$A_n \;=\; \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx \;=\; \frac{4 - 4\cos n\pi - 2n\pi \sin n\pi}{n^3\pi} \;=\; \begin{cases} 8/n^3\pi & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

This gives the solution

$$y(x,t) \;=\; \frac{8}{\pi}\sum_{n \text{ odd}} \frac{\cos 10nt \sin nx}{n^3}.$$

7. Here $a = 10$ and $L = 1.$ To satisfy the condition $y(x,\, 0) = 0$ we choose $A_n = 0$ for all $n,$ so

$$y(x,\,t) \;=\; \sum_{n=1}^{\infty} B_n \sin 10n\pi t \sin n\pi x.$$

To satisfy the condition $y_t(x, 0) = x$ we choose

$$B_n \;=\; \frac{1}{10n\pi}\cdot\frac{2(-1)^{n+1}}{n\pi} \;=\; \frac{(-1)^{n+1}}{5n^2\pi^2}$$

for $n \ge 1$ (see Equation (16) in Section 9.3). This gives

$$y(x,\,t) \;=\; \frac{1}{5\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sin 10n\pi t \sin n\pi x.$$

8. Here $a = 2$ and $L = \pi$. To satisfy the condition $y(x, 0) = \sin x$ we choose $A_1 = 1$ and $A_n = 0$ for $n > 1$, so

$$y(x,\,t) \;=\; \cos 2t \sin x + \sum_{n=1}^{\infty} B_n \sin 2nt \sin nx, \text{ so}$$

$$y_t(x,\,t) \;=\; -2 \sin 2t \sin x + \sum_{n=1}^{\infty} 2nB_n \cos 2nt \sin nx.$$

The condition $y_t(x, 0) = 1$ will be satisfied if $2nB_n = 4/\pi n$ for n odd and $b_n = 0$ for n even. We therefore choose $B_n = 2/\pi n^2$ for n odd and $B_n = 0$ for n even, so

$$y(x,t) \;=\; \cos 2t \sin x + \frac{4}{\pi}\sum_{n \text{ odd}}\frac{\sin 2nt \sin nx}{n}.$$

9. Here $a = 2$ and $L = 1$. To satisfy the condition $y(x, 0) = 0$ we choose $A_n = 0$ for all n, so

$$y(x,\,t) \;=\; \sum_{n=1}^{\infty} B_n \sin 2n\pi t \sin n\pi x.$$

To satisfy the condition $y_t(x, 0) = x(1-x)$ we choose

$$B_n \;=\; \frac{1}{n\pi}\int_0^1 x(1-x)\sin n\pi x\,dx \;=\; \frac{2-2\cos n\pi - n\pi \sin n\pi}{n^4\pi^4}.$$

Hence

$$y(x,t) \;=\; \frac{4}{\pi^4}\sum_{n \text{ odd}}\frac{\sin 2n\pi t \sin n\pi x}{n^4}.$$

10. Here $a = 5$ and $L = \pi$ so

$$y(x,\,t) \;=\; \sum_{n=1}^{\infty}\left(A_n \cos 5nt + B_n \sin 5nt\right)\sin nx.$$

We first compute the Fourier sine series $\sin^2 x = \sum_{n=1}^{\infty} b_n \sin nx$ and find that $b_n = 0$ if n is even whereas

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin nx\,dx = \frac{4(\cos n\pi - 1)}{\pi n(n^2-4)} = \frac{8}{\pi n(4-n^2)}$$

if n is odd. To satisfy the condition $y(x, t) = \sin^2 x$ we choose $A_n = b_n$, and to satisfy the condition $y_t(x, t) = \sin^2 x$ we choose $B_n = b_n/5n$. Then

$$y(x,t) = \frac{8}{5\pi}\sum_{n\text{ odd}} \frac{(5n\cos 5nt + \sin 5nt)\sin nx}{n^2(4-n^2)}.$$

11. Substitution of $L = 2$ ft, $T = 32$ lb, and the *linear* density

$$\rho = \frac{1/32\text{ oz}}{2\text{ ft}} = \frac{1\text{ oz}}{64\text{ ft}}\cdot\frac{1\text{ lb}}{16\text{ oz}}\cdot\frac{1\text{ slug}}{32\text{ lb}} = \frac{1}{32^3}\frac{\text{slug}}{\text{ft}}$$

in Eqs. (2) and (26) in the text yields the velocity $a = \sqrt{T/\rho} = \sqrt{32^4} = 1024\text{ ft/sec}$ with which waves move along the string, and its fundamental frequency

$$\nu_1 = \frac{1}{2L}\sqrt{\frac{T}{\rho}} = \frac{a}{2L} = 256\text{ Hz},$$

which is approximately middle C.

12. The value of

$$y(x,t) = \frac{4v_0 L}{\pi^2 a}\sum_{n\text{ odd}} \frac{1}{n^2}\sin\frac{n\pi a t}{L}\sin\frac{n\pi x}{L}$$

is maximal when each of the sine products is 1. This happens when $x = L/2,\ t = L/2a$:

$$y\left(\frac{L}{2},\frac{L}{2a}\right) = \frac{4v_0 L}{\pi^2 a}\sum_{n\text{ odd}} \frac{1}{n^2}\sin^2\frac{n\pi}{2} = \frac{4v_0 L}{\pi^2 a}\sum_{n\text{ odd}} \frac{1}{n^2} = \frac{4v_0 L}{\pi^2 a}\cdot\frac{\pi^2}{8} = \frac{v_0 L}{2a}.$$

Using fps units with the string of Problem 11 where $L = 2$ ft, $a = 1024$ ft/sec, and $v_0 = 60$ mph $= 88$ ft/sec, we get

$$y_{\max} = \frac{88\times 2}{2\times 1024} \approx 0.0859\text{ ft} \approx 1\text{ inch}.$$

13. If $y(x,t) = F(x+at) = F(u)$ with $u = x + at$, then the chain rule gives

$$\frac{\partial y}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x} = F'(u)\cdot 1 = F'(x+at);$$
$$\frac{\partial y}{\partial t} = \frac{dF}{du}\frac{\partial u}{\partial t} = F'(u)\cdot a = a\,F'(x+at) = a\frac{\partial y}{\partial x};$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{dF'}{du}\frac{\partial u}{\partial x} = F''(u)\cdot 1 = F''(x+at);$$
$$\frac{\partial^2 y}{\partial t^2} = a\frac{dF'}{du}\frac{\partial u}{\partial t} = a\cdot F''(u)\cdot a = a^2F''(x+at) = a^2\frac{\partial^2 y}{\partial x^2}.$$

14.
$$y(0,t) = \tfrac{1}{2}[F(at)+F(-at)] = \tfrac{1}{2}[F(at)-F(at)] = 0$$
$$y(L,t) = \tfrac{1}{2}[F(L+at)+F(L-at)]$$
$$= \tfrac{1}{2}[F(L+at)-F(-L+at)] = \tfrac{1}{2}[F(2L+(-L+at))-F(-L+at)] = 0$$
$$y(x,0) = \tfrac{1}{2}[F(x)+F(x)] = F(x)$$
$$y_t(x,t) = \tfrac{1}{2}[aF'(x+at)-aF'(x-at)]$$
$$y_t(x,0) = \tfrac{1}{2}[aF'(x)-aF'(x)] = 0$$

15. If $y(x,0)=0$ then the fundamental theorem of calculus gives

$$y(x,t) = y(x,t)-y(x,0) = \int_0^t y_t(x,\tau)\,d\tau = \int_0^t \frac{1}{2}\big[G(x+a\tau)+G(x-a\tau)\big]\,d\tau.$$

16. If $u=x+at,\ v=x-at$ then we solve readily for $x=\tfrac{1}{2}(u+v), t=\tfrac{1}{2a}(u-v)$. Hence

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\Big[y\big(\tfrac{1}{2}(u+v),\tfrac{1}{2a}(u-v)\big)\Big] = \frac{\partial y}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial y}{\partial t}\frac{\partial t}{\partial u} = \frac{1}{2}\frac{\partial y}{\partial x}+\frac{1}{2a}\frac{\partial y}{\partial t};$$
$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\Big[y\big(\tfrac{1}{2}(u+v),\tfrac{1}{2a}(u-v)\big)\Big] = \frac{\partial y}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial t}\frac{\partial t}{\partial v} = \frac{1}{2}\frac{\partial y}{\partial x}-\frac{1}{2a}\frac{\partial y}{\partial t};$$
$$\frac{\partial^2 y}{\partial v\partial u} = \frac{\partial}{\partial v}\left(\frac{1}{2}\frac{\partial y}{\partial x}+\frac{1}{2a}\frac{\partial y}{\partial t}\right)$$
$$= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial y}{\partial x}+\frac{1}{2a}\frac{\partial y}{\partial t}\right)-\frac{1}{2a}\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\partial y}{\partial x}+\frac{1}{2a}\frac{\partial y}{\partial t}\right)$$
$$= \frac{1}{4}\frac{\partial^2 y}{\partial x^2}+\frac{1}{4a}\frac{\partial^2 y}{\partial x\partial t}-\frac{1}{4a}\frac{\partial^2 y}{\partial t\partial x}-\frac{1}{4a^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{4a^2}\left(\frac{\partial^2 y}{\partial x^2}-a^2\frac{\partial^2 y}{\partial t^2}\right) = 0.$$

Now if $\dfrac{\partial^2 y}{\partial v\partial u} = \dfrac{\partial}{\partial v}\left(\dfrac{\partial y}{\partial u}\right)=0$ then antidifferentiation with respect to v gives

$\partial y/\partial u = G(v)$, an arbitrary function of v. Finally, antidifferentiation with respect to u gives $y = F(u) + G(v) = F(x+at) + G(x-at)$.

18. When we separate variables as in Equations (8)–(12) in this section, we find that $X(x)$ must satisfy the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(L) = 0.$$

In Example 4 of Section 3.8 we found that the eigenvalues and eigenfunctions of this problem are

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \qquad X_n(x) = \sin\frac{(2n-1)\pi x}{2L}$$

for $n = 1, 2, 3, \cdots$. The function $T_n(t)$ must satisfy the conditions

$$T_n'' + \lambda_n a^2 T_n = 0, \qquad T_n'(0) = 0,$$

so it follows that

$$T_n(t) = \cos\frac{(2n-1)\pi at}{2L}.$$

Thus the form of $y(t)$ is

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi at}{2L}\sin\frac{(2n-1)\pi x}{2L}.$$

Finally, in order to satisfy the initial condition $y(x, 0) = f(x)$ we use the odd half-multiple sine series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{(2n-1)\pi x}{2L}$$

discussed in Problem 21 of Section 9.3.

19. The general solution of the second-order ordinary differential equation $a^2 y'' = g$ is a second-order polynomial in x with leading coefficient $g/2a^2$. But the polynomial $\phi(x) = gx(x-L)/2a^2$ has this leading coefficient and satisfies the endpoint conditions $y(0) = y(L) = 0$.

20. If $y(x,t) = v(x,t) + \phi(x)$, then $y_{tt} = v_{tt}$ and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \phi''(x) = \frac{\partial^2 v}{\partial x^2} + \frac{g}{a^2}, \quad \text{so} \quad a^2\frac{\partial^2 y}{\partial x^2} - g = a^2\frac{\partial^2 v}{\partial x^2}.$$

The transformation of the boundary conditions is straightforward.

22. The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X(L) = 0$$

has the usual eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad X_n(x) = \sin\frac{n\pi x}{L}$$

for $n = 1, 2, 3, \cdots$. The function $T_n(t)$ satisfies the equation

$$T_n'' + \omega_n^2 R\, T_n = 0, \qquad \omega_n^2 = \frac{n^2\pi^2 a^2}{L^2} - h^2 > 0,$$

so

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t.$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\omega_n t + B_n \sin\omega_n t\right)\sin\frac{n\pi x}{L}.$$

To satisfy the conditions $v(x, 0) = f(x)$ and $v_t(x, 0) = h\,f(x)$ we choose $A_n = b_n$ and $B_n = hb_n/\omega_n$ where

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

Then

$$v(x,t) = \sum_{n=1}^{\infty}\frac{b_n}{\omega_n}\left(\omega_n \cos\omega_n t + h\sin\omega_n t\right)\sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} c_n \cos(\omega_n t - \alpha_n)\sin\frac{n\pi x}{L},$$

where

$$c_n = \frac{b_n}{\cos\alpha_n} \quad \text{and} \quad \alpha_n = \tan^{-1}\frac{h}{\omega_n}.$$

Finally,

$$y(x,t) = e^{-ht}v(x,t) = e^{-ht}\sum_{n=1}^{\infty} c_n \cos(\omega_n t - \alpha_n)\sin\frac{n\pi x}{L}.$$

23. First, if $0 \le x \le \frac{\pi}{4}$, then $\frac{\pi}{4} \le x + \frac{\pi}{4} \le \frac{\pi}{2}$ and $-\frac{\pi}{4} \le x - \frac{\pi}{4} \le 0$, so that

$$\begin{aligned} y\left(x,\frac{\pi}{4}\right) &= \frac{1}{2}\left[F\left(x+\frac{\pi}{4}\right)+F\left(x-\frac{\pi}{4}\right)\right] \\ &= \frac{1}{2}\left[1-\cos 2\left(x+\frac{\pi}{4}\right)+\cos 2\left(x-\frac{\pi}{4}\right)-1\right] \\ &= \frac{1}{2}\left[\not{1}-\cos\left(2x+\frac{\pi}{2}\right)+\cos\left(2x-\frac{\pi}{2}\right)-\not{1}\right] \\ &= \frac{1}{2}(\sin 2x+\sin 2x) \\ &= \sin 2x. \end{aligned}$$

Next, if $\frac{\pi}{4} \le x \le \frac{3\pi}{4}$, then $\frac{\pi}{2} \le x+\frac{\pi}{4} \le \pi$ and $0 \le x-\frac{\pi}{4} \le \frac{\pi}{2}$, so that

$$\begin{aligned} y\left(x,\frac{\pi}{4}\right) &= \frac{1}{2}\left[F\left(x+\frac{\pi}{4}\right)+F\left(x-\frac{\pi}{4}\right)\right] \\ &= \frac{1}{2}\left[1-\cos 2\left(x+\frac{\pi}{4}\right)+1-\cos 2\left(x-\frac{\pi}{4}\right)\right] \\ &= \frac{1}{2}\left[1-\cos\left(2x+\frac{\pi}{2}\right)+1-\cos\left(2x-\frac{\pi}{2}\right)\right] \\ &= \frac{1}{2}\left(2+\cancel{\sin 2x}-\cancel{\sin 2x}\right) \\ &= 1. \end{aligned}$$

Finally, if $\frac{3\pi}{4} \le x \le \pi$, then $\pi \le x+\frac{\pi}{4} \le \frac{5\pi}{4}$ and $\frac{\pi}{2} \le x-\frac{\pi}{4} \le \frac{3\pi}{4}$, so that

$$\begin{aligned} y\left(x,\frac{\pi}{4}\right) &= \frac{1}{2}\left[F\left(x+\frac{\pi}{4}\right)+F\left(x-\frac{\pi}{4}\right)\right] \\ &= \frac{1}{2}\left[\cos 2\left(x+\frac{\pi}{4}\right)-\not{1}+\not{1}-\cos 2\left(x-\frac{\pi}{4}\right)\right] \\ &= \frac{1}{2}\left[\cos\left(2x+\frac{\pi}{2}\right)-\cos\left(2x-\frac{\pi}{2}\right)\right] \\ &= \frac{1}{2}(-\sin 2x-\sin 2x) \\ &= -\sin 2x. \end{aligned}$$

24. **(a)** $f''(x) = 4\cos 2x = 0$ if $x = \pi/4$ or $x = 3\pi/4$.

(b) If $0 \le t \le \pi/4$ then $0 \le \pi/4 \pm t \le \pi/2$ so

$$\begin{aligned} y(\pi/4, t) &= \frac{1}{2}[F(\pi/4 + t) + F(\pi/4 - t)] \\ &= \frac{1}{2}[1 - \cos 2(\pi/4 + t) + 1 - \cos 2(\pi/4 - t)] \\ &= \frac{1}{2}[1 - \cos(\pi/2 + 2t) + 1 - \cos(\pi/2 - 2t)] \\ &= \frac{1}{2}[2 + \sin 2t - \sin 2t] \\ y(\pi/4, t) &= 1 \end{aligned}$$

SECTION 9.7

STEADY-STATE TEMPERATURE AND LAPLACE′S EQUATION

1. Because $Y(0) = Y(b) = 0$ we take our separation of variables in the form

$$X'' - \lambda X = 0 = Y'' + \lambda Y$$

with $\lambda > 0$. Then it follows that

$$Y_n(y) = \sin\frac{n\pi y}{b}, \qquad \lambda_n = \frac{n^2\pi^2}{b^2}$$

and thence that

$$X_n(x) = A_n \cosh\frac{n\pi x}{b} + B_n \sinh\frac{n\pi x}{b}.$$

The condition that $X(0) = 0$ implies that $A_n = 0$ so $X_n(x) = B_n \cosh n\pi x/b$, and hence

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}.$$

Finally we satisfy the condition $u(a, y) = g(y)$ by choosing $C_n = b_n/(\sinh n\pi a/b)$, where the $\{b_n\}$ are the Fourier sine coefficients of $g(y)$ on $0 \le y \le b$.

2. Because $Y(0) = Y(b) = 0$ we take our separation of variables in the form

$$X'' - \lambda X = 0 = Y'' + \lambda Y$$

with $\lambda > 0$. Then it follows that

$$Y_n(y) = \sin\frac{n\pi y}{b}, \qquad \lambda_n = \frac{n^2\pi^2}{b^2}$$

and thence that

$$X_n(x) = A_n \cosh\frac{n\pi x}{b} + B_n \sinh\frac{n\pi x}{b}.$$

The condition $X(a) = 0$ implies that

$$B_n = -\frac{A_n \cosh n\pi a/b}{\sinh n\pi a/b}.$$

It now follows as in Equation (12) in the text that

$$X_n(x) = C_n \sinh\frac{n\pi(a-x)}{b},$$

so

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi(a-x)}{b}\sin\frac{n\pi y}{b}.$$

Finally we satisfy the condition $u(0, y) = g(y)$ by choosing $C_n = b_n/(\sinh n\pi a/b)$, where the $\{b_n\}$ are the Fourier sine coefficients of $g(y)$ on $0 \le y \le b$.

3. Just as in Example 1 of Section 9.7 we have $X_n(x) = \sin n\pi x/a$ and

$$Y_n(y) = A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a}.$$

The condition $Y(0) = 0$ now yields $A_n = 0$ so $Y_n(y) = B_n \sinh n\pi y/a$, and hence

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}.$$

Finally we satisfy the condition $u(x, b) = f(x)$ by choosing $C_n = b_n/(\sinh n\pi b/a)$, where the $\{b_n\}$ are the Fourier sine coefficients of $f(x)$ on $0 \le x \le a$.

4. Because $X'(0) = X'(a) = 0$, we work with the separation of variables

$$X'' + \lambda X = 0 = Y'' - \lambda Y.$$

The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X'(0) = X'(a) = 0$$

has eigenvalues and eigenfunctions $\lambda_0 = 0$, $X_0(x) = 1$ and

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \qquad X_n(x) = \cos\frac{n\pi x}{a}$$

for $n = 1, 2, 3, \cdots$. When $n = 0$, $Y_0'' = 0$ yields $Y_0(y) = Ay + B$. Then $Y_0(0) = 0$

gives $B = 0$, so we take $Y_0(y) = y$. For $n > 0$ we have

$$Y_n(y) = A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a},$$

and $Y_n(0) = 0$ gives $A_n = 0$, so

$$u(x,y) = B_0 y + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}.$$

Finally

$$u(x,b) = B_0 b + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{a} \sinh\frac{n\pi b}{a},$$

so we satisfy the condition $u(x, b) = f(x)$ by taking $B_0 = a_0/2b$ and $B_n = a_n \Big/ \left(\sinh\frac{n\pi b}{a}\right)$, where $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a}$.

5. Now $Y'(0) = Y'(b) = 0$, so we work with the separation of variables

$$X'' - \lambda X = 0 = Y'' + \lambda Y.$$

The eigenvalue problem

$$Y'' + \lambda Y = 0, \qquad Y'(0) = Y'(b) = 0,$$

has eigenvalues and eigenfunctions $\lambda_0 = 0$, $Y_0(y) = 1$ and

$$\lambda_n = \frac{n^2\pi^2}{b^2}, \qquad Y_n(y) = \cos\frac{n\pi y}{b}$$

for $n = 1, 2, 3, \cdots$. When $n = 0$, $X_0''(x) \equiv 0$ yields $X_0(x) = Ax + B$. Then $X_0(a) = 0$ is satisfied by $X_0(x) = a - x$. For $n > 0$ we have

$$X_n(x) = A_n \cosh\frac{n\pi x}{b} + B_n \sinh\frac{n\pi x}{b},$$

and $X_n(a) = 0$ is satisfied by the particular linear combination

$$X_n(x) = C_n \sinh\frac{n\pi(a-x)}{b},$$

of $\cosh n\pi x/b$ and $\sinh n\pi x/b$. Therefore

$$u(x,y) = C_0(a-x) + \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi(a-x)}{b} \cos\frac{n\pi y}{b}.$$

Finally we satisfy the condition $u(0, y) = g(y)$ by choosing

$$C_0 = \frac{a_0}{2a} \qquad \text{and} \qquad C_n = \frac{b_n}{\sinh n\pi a/b},$$

where the $\{a_n\}$ are the Fourier cosine coefficients of $g(y)$ on $0 \le y \le b$.

6. This is the same as Problem 4 except that $Y'(0) = 0$ instead of $Y(0) = 0$, so $Y_0(y) = 1$ and $Y_n(y) = A_n \cosh n\pi y/a$ for $n > 0$. Then

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a},$$

so we satisfy the condition $u(x, b) = f(x)$ by choosing $A_0 = a_0/2$ and $A_n = a_n/(\cosh n\pi b/a)$, where $\{a_n\}$ are the Fourier cosine coefficients of $f(x)$ on $[0, a]$.

7. The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X(a) = 0$$

yields the eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \qquad X_n(x) = \sin\frac{n\pi x}{a}$$

for $n = 1, 2, 3, \cdots$. Then

$$Y_n'' + \lambda_n Y_n = 0$$

yields

$$Y_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}.$$

In order that $Y(y) \to 0$ as $y \to \infty$ we take $A_n = 0$, so

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-n\pi y/a} \sin\frac{n\pi x}{a}.$$

Finally we satisfy the condition $u(x,0) = f(x)$ by choosing the constants $\{B_n\}$ as the Fourier sine coefficients of $f(x)$ on $0 \le x \le a$.

8. The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X'(0) = X'(a) = 0$$

yields $\lambda_0 = 0$, $X_0(x) = 1$ and

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \qquad X_n(x) = \cos\frac{n\pi x}{a}$$

for $n > 0$. Then

$$Y_n'' + \lambda_n Y_n = 0$$

yields $Y_0(y) = A_0 y + B_0$ and

$$Y_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}.$$

In order that $Y(y)$ be bounded as $y \to \infty$, we take $A_0 = 0$ and $A_n = 0$ for $n > 0$, so

$$u(x,y) = B_0 + \sum_{n=1}^{\infty} B_n e^{-n\pi y/a} \cos\frac{n\pi x}{a}.$$

Finally we satisfy the condition $u(x, 0) = f(x)$ by choosing $B_0 = a_0/2$ and $B_n = a_n$ where the $\{a_n\}$ are the Fourier cosine coefficients of $f(x)$ on $[0, a]$.

9. If in Problem 8 we have $f(x) = 10x$ on $0 < x < 10$, then

$$a_0 = \frac{2}{10}\int_0^{10} 10x\,dx = 100,$$

$$a_n = \frac{2}{10}\int_0^{10} 10x\cos\frac{n\pi x}{10}\,dx = \frac{200(\cos n\pi - 1 + n\pi\sin\pi)}{n^2\pi^2},$$

so

$$u(x,y) = 50 - \frac{400}{\pi^2}\sum_{n\text{ odd}} \frac{1}{n^2} e^{-n\pi y/10}\cos\frac{n\pi x}{10}.$$

Then

$$u(0,5) \approx 50 - 8.4250 - 0.0405 - 0.0006 - 0.0000 - \cdots \approx 41.53,$$

$$u(5,5) = 50 - 0 - 0 - 0 - 0 - \cdots = 50,$$

$$u(0,5) \approx 50 + 8.4250 + 0.0405 + 0.0006 + 0.0000 + \cdots \approx 58.47.$$

10. The boundary value problem is

$$u_{xx} + u_{yy} = 0 \qquad (0 < x < a,\ 0 < y < b)$$

$$u(0, y) = u_x(a, y) = u(x, 0) = 0,$$

$$u(x, b) = f(x).$$

The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(a) = 0$$

yields (by Example 4 in Section 3.8)

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4a^2}, \qquad X_n(x) = \sin\frac{(2n-1)\pi x}{2a}$$

for $n = 1, 2, 3, \cdots$. Then

$$Y_n'' - \lambda_n Y_n = 0$$

yields

$$Y_n(y) = A_n \cosh\frac{(2n-1)\pi y}{2a} + B_n \sinh\frac{(2n-1)\pi y}{2a}.$$

Because $Y(0) = 0$, we choose $A_n = 0$, so

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2a} \sinh\frac{(2n-1)\pi y}{2a}.$$

Finally we satisfy the condition $u(x, b) = f(x)$ by choosing

$$B_n = \frac{b_{2n-1}}{\sinh[(2n-1)\pi b/2a]},$$

where the $\{b_{2n-1}\}$ are the odd half-multiple sine coefficients of $f(x)$ on $[0, a]$, as given by Problem 21 in Section 9.3.

11. Now the boundary value problem is

$$u_{xx} + u_{yy} = 0 \qquad (0 < x < a,\ 0 < y < b)$$
$$u(a, y) = u_y(x, 0) = u(x, b) = 0,$$
$$u(0, y) = g(y).$$

The eigenvalue problem

$$Y'' + \lambda Y = 0, \qquad Y'(0) = Y(b) = 0$$

yields (similar to Example 4 in Section 3.8)

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4b^2}, \qquad Y_n(y) = \cos\frac{(2n-1)\pi y}{2b}$$

for $n = 1, 2, 3, \cdots$. Then

$$X_n'' - \lambda_n X_n = 0$$

yields

$$X_n(x) = A_n \cosh\frac{(2n-1)\pi x}{2b} + B_n \sin\frac{(2n-1)\pi x}{2b}.$$

Now $X_n(a) = 0$ is satisfied by the particular linear combination

$$X_n(x) = C_n \sinh\frac{(2n-1)\pi(a-x)}{2b}$$

of $\cosh(2n-1)\pi x/2b$ and $\sinh(2n-1)\pi x/2b$. Hence

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\frac{(2n-1)\pi(a-x)}{2b}\cos\frac{(2n-1)\pi y}{2b}$$
$$= \sum_{n \text{ odd}} A_n \sinh\frac{n\pi(a-x)}{2b}\cos\frac{n\pi y}{2b}.$$

Finally we satisfy the condition $u(0, y) = g(y)$ by choosing

$$A_n = \frac{a_n}{\sinh n\pi a/2b},$$

where the $\{a_n\}$ are the odd half-multiple cosine coefficients of $g(y)$ on $[0, b]$, as given by Problem 22 in Section 9.3.

12. The boundary value problem is

$$u_{xx} + u_{yy} = 0 \qquad (0 < x < 30,\ y > 0)$$
$$u(0, y) = u_x(30, y) = 0$$
$$u(x, y) \text{ bounded as } y \to \infty$$
$$u(x, 0) = 25$$

The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(30) = 0$$

yields (by Example 4 in Section 3.8)

$$\lambda_n = \frac{(2n-1)^2\pi^2}{3600}, \qquad X_n(x) = \sin\frac{(2n-1)\pi x}{60}$$

for $n = 1, 2, 3, \cdots$. Then

$$Y_n'' - \lambda_n Y_n = 0$$

yields

$$Y_n(y) = A_n \exp\left[\frac{(2n-1)\pi y}{60}\right] + B_n \exp\left[-\frac{(2n-1)\pi y}{60}\right].$$

and we take $A_n = 0$ in order that $Y_n(y)$ be bounded as $y \to \infty$. Hence

$$u(x,y) = \sum_{n \text{ odd}} b_n e^{-n\pi y/60} \sin\frac{n\pi x}{60}.$$

Finally, by Problem 21 in Section 9.3, the odd half-multiple Fourier sine coefficients of $u(x, 0) = 25$ on $[0, 30]$ are given by

$$b_n = \frac{2}{30}\int_0^{30} 25\sin\frac{n\pi x}{60}\,dx = \frac{200}{n\pi}\sin^2\frac{n\pi}{4} = \frac{100}{n\pi}$$

for n odd. Thus

$$u(x,y) = \frac{100}{\pi}\sum_{n\text{ odd}}\frac{1}{n}e^{-n\pi y/60}\sin\frac{n\pi x}{60}.$$

13. We start with the periodic polar-coordinate solution

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta)$$

and choose $a_n \equiv 0$ in order to satisfy the conditions $u(r,0) = u(r,\pi) = 0$. Then

$$u(r,\theta) = \sum_{n=1}^{\infty} r^n c_n \sin n\theta$$

satisfies the nonhomogeneous boundary condition $u(a,\theta) = f(\theta)$ provided that $a^n c_n$ is is the nth Fourier sine coefficient of $f(\theta)$ on the interval $0 < \theta < \pi$, that is,

$$c_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta.$$

14. We start with the periodic polar-coordinate solution

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta)$$

and choose $b_n \equiv 0$ in order to satisfy the conditions $u_\theta(r,0) = u_\theta(r,\pi) = 0$. Then

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n c_n \cos n\theta$$

satisfies the nonhomogeneous boundary condition $u(a,\theta) = f(\theta)$ provided that $a^n c_n$ is is the nth Fourier sine coefficient of $f(\theta)$ on the interval $0 < \theta < \pi$, that is,

$$c_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\cos n\theta\,d\theta.$$

15. As in the textbook discussion of the polar-coordinate Dirichlet problem, the substitution $u(r,\theta) = R(r)\Theta(\theta)$ in Laplace's equation yields the separated ordinary differential equations

$$r^2R'' + rR' - \lambda R = 0 \tag{25}$$

and

$$\Theta'' + \lambda\Theta = 0. \tag{26}$$

With $\lambda = \alpha^2$ the general solution of (26) is

$$\Theta(\theta) = A\cos\alpha\theta + B\sin\alpha\theta,$$

and the endpoint condition $\Theta(0) = \Theta'(0) = 0$ yields $A = 0$ and $\theta = (2n-1)/2,$ so the nth eigenvalue and eigenfunction are given by

$$\lambda_n = \frac{(2n-1)^2}{4}, \qquad \Theta_n(\theta) = \sin\frac{(2n-1)\theta}{2}.$$

As in the discussion of Eqs. (29) and (30) in the text, the bounded solution of

$$r^2R_n'' + rR_n' - \frac{(2n-1)^2}{4}R_n = 0$$

is

$$R_n(r) = r^{(2n-1)/2}$$

for $n = 1, 2, 3, \cdots$. We thereby obtain the formal series solution

$$u(r,\theta) = \sum_{n\text{ odd}} c_n r^{n/2}\sin\frac{n\theta}{2}.$$

It remains only to satisfy the nonhomogeneous boundary condition $u(a,\theta) = f(\theta)$ by choosing

$$c_n = \frac{2}{\pi a^{n/2}}\int_0^\pi f(\theta)\sin\frac{n\theta}{2}\,d\theta,$$

so that (for n odd) $c_n a^{n/2}$ equals the nth odd half-multiple sine coefficient of $f(\theta)$.

16. The only difference between the exterior problem here and the interior problem in the text is that in

$$R_n(r) = C_n r^n + D_n r^{-n}$$

we must choose $C_n = 0$ in order that $R_n(r)$ be bounded as $r \to \infty$.

17. The substitution $u(r,\theta) = R(r)\Theta(\theta)$ in Laplace's equation yields the same separated solution functions

$$\Theta_0(\theta) = 1, \qquad R_0(r) = C_0 + D_0\ln r$$

and

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta, \qquad R_n(r) = C_n r^n + \frac{D_n}{r^n}$$

as in Eqs. (28)-(30) in the text. We choose $B_n \equiv 0$ to satisfy the boundary condition $u(r,\theta) = u(r,-\theta)$, and $n = 1$ with $C_1 = U_0$ to satisfy the given limit condition as

$r \to \infty$. Then the condition that $u_r(a,\theta) = 0$ requires that $D_1 = U_0 a^2$, so

$$u(r,\theta) = \frac{U_0}{r}\left(r^2 + a^2\right)\cos\theta.$$

20. When we substitute $v(r, t) = r\,u(r, t)$ we get the boundary value problem

$$v_t = kv_{rr} \qquad (r < a, \quad t > 0)$$
$$v(0, t) = v(a, t) = 0$$
$$v(r, 0) = T_0 r$$

that corresponds to a heated rod along the interval $0 \le r \le a$. It therefore follows from Equation (31) in Section 9.5 that

$$v(r,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 kt}{a^2}\right)\sin\frac{n\pi x}{a}.$$

To get the formula given in the text it remains only to calculate the Fourier sine coefficients $\{b_n\}$ of $f(r) = T_0 r$ on $0 < r < a$, and finally to divide $v(r, t)$ by r to get $u(r, t)$.

21. **(a)** Since we cannot simply substitute $r = 0$, we apply continuity of $u(r, t)$ at $r = 0$ and calculate

$$u(0,t) = \lim_{r\to 0} u(r,t)$$

noting that

$$\lim_{r\to 0}\frac{\sin n\pi r/a}{r} = \frac{n\pi}{a}\lim_{r\to 0}\frac{\sin n\pi r/a}{n\pi r/a} = \frac{n\pi}{a}\lim_{r\to 0}\frac{\sin\theta}{\theta} = \frac{n\pi}{a}$$

by the elementary fact that $(\sin\theta)/\theta \to 1$ as $\theta \to 0$.

(b) With $a = 30$ and $T_0 = 100$ we have

$$u(0,t) = 200\sum_{n=1}^{\infty}(-1)^{n+1}\exp\left(-\frac{n^2\pi^2 kt}{900}\right).$$

If $k = 0.15$ for iron then after 15 minutes = 900 seconds the center temperature is

$$u(0,900) \approx 45.5075 - 0.5361 + 0.0003 - 0.0000 + \cdots \approx 44.97.$$

If $k = 0.005$ for iron then after 15 minutes the center temperature is

$$\begin{aligned} u(0,900) &\approx 190.37-164.174+128.276-90.8081+58.2426 \\ &\quad -33.8449+17.8190-8.4998+3.6734-1.4384 \\ &\quad +0.5103-0.1640+0.0478-0.0126+0.0030 \\ &\quad -0.0007+0.0001-0.00002+0.00000-\cdots \\ u(0,900) &\approx 100.00 \end{aligned}$$

Thus the center of the ball has not yet begun to cool. For the center of this concrete ball to reach 45° (as with the iron ball after 15 minutes) would require $(0.15/0.005)\times 15$ = 450 minutes, that is, seven and a half hours!