21-1.

Show that the sum of the moments of inertia of a body, $I_{xx} + I_{yy} + I_{zz}$, is independent of the orientation of the $x, y, z$ axes and thus depends only on the location of its origin.

**SOLUTION**

\[
I_{xx} + I_{yy} + I_{zz} = \int m (y^2 + z^2)dm + \int m (x^2 + z^2)dm + \int m (x^2 + y^2)dm
\]

\[
= 2 \int_m (x^2 + y^2 + z^2)dm
\]

However, $x^2 + y^2 + z^2 = r^2$, where $r$ is the distance from the origin $O$ to $dm$. Since $|r|$ is constant, it does not depend on the orientation of the $x, y, z$ axis. Consequently, $I_{xx} + I_{yy} + I_{zz}$ is also independent of the orientation of the $x, y, z$ axis. Q.E.D.
21–2.

Determine the moment of inertia of the cone with respect to a vertical \( y \) axis passing through the cone’s center of mass. What is the moment of inertia about a parallel axis \( y' \) that passes through the diameter of the base of the cone? The cone has a mass \( m \).

**SOLUTION**

The mass of the differential element is \( dm = \rho dV = \rho (\pi y^2) dx = \frac{\rho \pi a^2}{h^2} x^2 dx \).

\[
dI_y = \frac{1}{4} (dm)^2 + (dx)^2
\]

\[
= \frac{1}{4} \left[ \frac{\rho \pi a^2}{h^2} x^2 dx \right]^2 + \left( \frac{\rho \pi a^2}{h^2} x \right)^2 dx
\]

\[
= \frac{\rho \pi a^2}{4h^4} (4h^2 + a^2) x^2 dx
\]

\[
I_y = \int dI_y = \frac{\rho \pi a^2}{4h^4} (4h^2 + a^2) \int_0^h x^4 dx = \frac{\rho \pi a^2 h}{20} (4h^2 + a^2)
\]

However,

\[
m = \int dm = \frac{\rho \pi a^2}{h^2} \int_0^h x^2 dx = \frac{\rho \pi a^2 h}{3}
\]

Hence,

\[
I_y = \frac{3m}{20} (4h^2 + a^2)
\]

Using the parallel axis theorem:

\[
I_y = I_\tau + md^2
\]

\[
= \frac{3m}{20} (4h^2 + a^2) = I_\tau + m \left( \frac{3h}{4} \right)^2
\]

\[
I_\tau = \frac{3m}{80} (h^2 + 4a^2)
\]

\[
I_y' = I_\tau + md^2
\]

\[
= \frac{3m}{80} (h^2 + 4a^2) + m \left( \frac{h}{4} \right)^2
\]

\[
= \frac{m}{20} (2h^2 + 3a^2)
\]
21–3.
Determine moment of inertia $I_y$ of the solid formed by revolving the shaded area around the $x$ axis. The density of the material is $\rho = 12$ slug/ft$^3$.

**SOLUTION**

The mass of the differential element is $dm = \rho dV = \rho (\pi y^2) \, dx = \rho \pi x \, dx$.

\[
dI_y = \frac{1}{4} dm y^2 + dm x^2
= \frac{1}{4} [\rho \pi x \, dx] (x) + (\rho \pi x \, dx) x^2
= \rho \pi \left( \frac{1}{4} x^2 + x^3 \right) \, dx
\]

\[
I_y = \int dI_y = \rho \pi \int_0^4 \left( \frac{1}{4} x^2 + x^3 \right) \, dx = 69.33 \pi \rho
= 69.33(\pi)(12) = 2614 \text{ slug} \cdot \text{ft}^2 \quad \text{Ans.}
\]
*21–4.

Determine the moments of inertia $I_x$ and $I_y$ of the paraboloid of revolution. The mass of the paraboloid is 20 slug.

**SOLUTION**

The mass of the differential element is $dm = \rho dV = \rho z^2 \, dy = 2 \rho \pi y \, dy$.

$$m = 20 = \int dm = \int_0^2 2 \rho \pi y \, dy$$

$$20 = 4 \rho \pi \quad \rho = \frac{4}{2} \text{ slug/ft}^3$$

$$dI_x = \frac{1}{4} dm \, z^2 + dm \left( y^2 \right)$$

$$= \frac{1}{4} [2 \rho \pi y \, dy] (2y) + [2 \rho \pi y \, dy] y^2$$

$$= \left( 5y^2 + 10y^3 \right) \, dy$$

$$I_x = \int dI_x = \int_0^2 \left( 5y^2 + 10y^3 \right) \, dy = 53.3 \text{ slug} \cdot \text{ft}^2$$

Ans.

$$dI_y = \frac{1}{4} dm \, z^2 = 2 \rho \pi y^2 \, dy$$

$$I_y = \int dI_y = \int_0^2 10y^2 \, dy = 26.7 \text{ slug} \cdot \text{ft}^2$$

Ans.

Ans:

$I_x = 53.3 \text{ slug} \cdot \text{ft}^2$

$I_y = 26.7 \text{ slug} \cdot \text{ft}^2$
21–5.

Determine by direct integration the product of inertia $I_{yz}$ for the homogeneous prism. The density of the material is $\rho$. Express the result in terms of the total mass $m$ of the prism.

**SOLUTION**

The mass of the differential element is $dm = \rho dV = \rho h dx dy = \rho (a - y) dy$.

$$m = \int dm = \rho h \int_{0}^{a} (a - y) dy = \frac{\rho a h}{2}$$

Using the parallel axis theorem:

$$dI_{yz} = (dI_{yz})_{G} + dmy_{G}z_{G}$$

$$= 0 + (\rho h dx dy) (y \left( \frac{h}{2} \right))$$

$$= \frac{\rho h^2}{2} xy dy$$

$$= \frac{\rho h^2}{2} (ay - y^2) dy$$

$$I_{yz} = \frac{\rho h^2}{2} \int_{0}^{a} (ay - y^2) dy = \frac{\rho a^2 h^3}{12} - \frac{1}{6} \left( \frac{\rho a^2 h^3}{2} \right)(ah) = \frac{m}{6} ah$$  \hspace{1cm} \text{Ans.}
21–6.

Determine by direct integration the product of inertia $I_{xy}$ for the homogeneous prism. The density of the material is $\rho$. Express the result in terms of the total mass $m$ of the prism.

**SOLUTION**

The mass of the differential element is $dm = \rho dV = \rho h dx dy = \rho h (a - y) dy$.

$$m = \int dm = \rho h \int_0^a (a - y) dy = \frac{\rho a^2 h}{2}$$

Using the parallel axis theorem:

$$dI_{xy} = (dI_{xy})_G + dm x_G y_G$$

$$= 0 + (\rho h x dy) \left( \frac{x}{2} \right) (y)$$

$$= \frac{\rho h}{2} x^2 y dy$$

$$= \frac{\rho h}{2} (y^3 - 2ay^2 + a^2 y) dy$$

$$I_{xy} = \frac{\rho h}{2} \int_0^a (y^3 - 2ay^2 + a^2 y) dy$$

$$= \frac{\rho a^4 h}{24} = \frac{1}{12} \left( \frac{\rho a^2 h}{2} \right) a^2 = \frac{m a^2}{12}$$

**Ans:**

$$I_{xy} = \frac{m a^2}{12}$$
21–7.

Determine the product of inertia $I_{xy}$ of the object formed by revolving the shaded area about the line $x = 5$ ft. Express the result in terms of the density of the material, $\rho$.

**SOLUTION**

$$
\int_0^3 dm = \rho 2\pi \int_0^3 (5 - x)y \, dx = \rho 2\pi \int_0^3 (5 - x)\sqrt{3x} \, dx = 38.4\rho \pi
$$

$$
\int_0^3 \bar{y} \, dm = \rho 2\pi \int_0^3 \frac{y}{2} (5 - x) \, dx
$$

$$
= \rho \pi \int_0^3 (5 - x)(3x) \, dx
$$

$$
= 40.5\rho \pi
$$

Thus, $\bar{y} = \frac{40.5\rho \pi}{38.4\rho \pi} = 1.055$ ft

The solid is symmetric about $y$, thus

$I_{xy} = 0$

$I_{xy} = I_{xy} + x\bar{y}m$

$$
= 0 + 5(1.055)(38.4\rho \pi)
$$

$I_{xy} = 636\rho$

**Ans:**

$I_{xy} = 636\rho$
**21–8.**

Determine the moment of inertia \( I_y \) of the object formed by revolving the shaded area about the line \( x = 5 \) ft. Express the result in terms of the density of the material, \( \rho \).

**SOLUTION**

\[
I_y = \int_0^3 \frac{1}{2} dm \, r^2 - \frac{1}{2}(m')(2)^2
\]

\[
\int_0^3 \frac{1}{2} dm \, r^2 = \frac{1}{2} \int_0^3 \rho \pi (5 - x)^4 \, dy
\]

\[
= \frac{1}{2} \rho \pi \int_0^3 (5 - \frac{y^2}{3})^4 \, dy
\]

\[
= 490.29 \, \rho \, \pi
\]

\[
m' = \rho \, \pi \, (2)^3(3) = 12 \, \rho \, \pi
\]

\[
I_y = 490.29 \, \rho \, \pi - \frac{1}{2} (12 \, \rho \, \pi)(2)^2 = 466.29 \, \rho \, \pi
\]

Mass of body;

\[
m = \int_0^3 \rho \, \pi (5 - x)^2 \, dy - m'
\]

\[
= \int_0^3 \rho \, \pi (5 - \frac{y^2}{3})^2 \, dy - 12 \, \rho \, \pi
\]

\[
= 38.4 \, \rho \, \pi
\]

\[
I_y = 466.29 \, \rho \, \pi + (38.4 \, \rho \, \pi)(5)^2
\]

\[
= 1426.29 \, \rho \, \pi
\]

\[
I_y = 4.48(10^3) \, \rho
\]

Ans.

Also,

\[
I_y = \int_0^3 r^2 \, dm
\]

\[
= \int_0^3 (5 - x)^2 \, \rho(2\pi)(5 - x) \, y \, dx
\]

\[
= 2 \, \rho \, \pi \int_0^3 (5 - x)^3(3x)^{1/2} \, dx
\]

\[
= 466.29 \, \rho \, \pi
\]

\[
m = \int_0^3 dm
\]

\[
= 2 \, \rho \, \pi \int_0^3 (5 - x) \, y \, dx
\]

\[
= 2 \, \rho \, \pi \int_0^3 (5 - x)(3x)^{1/2} \, dx
\]

\[
= 38.4 \, \rho \, \pi
\]

\[
I_y = 466.29 \, \rho \, \pi + 38.4 \, \rho \, \pi(5)^2 = 4.48(10^3)\rho
\]

Ans:

\[
I_y = 4.48(10^3) \, \rho
\]

\[
I_y = 4.48(10^3) \, \rho
\]
21–9.

Determine the moment of inertia of the cone about the $z'$ axis. The weight of the cone is 15 lb, the height is $h = 1.5$ ft, and the radius is $r = 0.5$ ft.

**SOLUTION**

\[ \theta = \tan^{-1}\left(\frac{0.5}{1.5}\right) = 18.43^\circ \]

\[ I_{xx} = I_{yy} = \frac{3}{80} m \left[ 4(0.5)^2 + (1.5)^2 \right] + m(1.5 - \frac{1.5}{4})^2 \]

\[ I_{zz} = I_{yy} = 1.3875 \ m \]

\[ I_z = \frac{3}{10} m(0.5)^2 = 0.075 \ m \]

\[ I_{xy} = I_{yz} = I_{zx} = 0 \]

Using Eq. 21–5.

\[ I_{z'} = u_{z'}^2 I_{xx} + u_{z'}^2 I_{yy} + u_{z'}^2 I_{zz} \]

\[ = 0 + [\cos(108.43^\circ)]^2(1.3875m) + [\cos(18.43^\circ)]^2(0.075m) \]

\[ = 0.2062 \ m \]

\[ I_{z'} = 0.2062 \left(\frac{15}{32.2}\right) = 0.0961 \text{ slug} \cdot \text{ft}^2 \]

**Ans:**

\[ I_{z'} = 0.0961 \text{ slug} \cdot \text{ft}^2 \]
21–10.
Determine the radii of gyration $k_x$ and $k_y$ for the solid formed by revolving the shaded area about the $y$ axis. The density of the material is $\rho$.

**SOLUTION**

For $k_y$:
The mass of the differential element is

$$dm = \rho dV = \rho(\pi x^2) dy = \rho \pi x^2 \frac{dy}{y^2}.$$  

$$dI_y = \frac{1}{2} dm x^2 = \frac{1}{2} \left( \rho \pi x^2 \frac{dy}{y^2} \right) (\frac{x}{y}) = \frac{1}{2} \rho \pi \frac{dy}{y}$$

$$I_y = \int dI_y = \frac{1}{2} \rho \pi \int \frac{dy}{y^2} + \frac{1}{2} \left( \rho (\pi x^2)(0.25) \right) (4)^2$$

$$m = \int dm = \rho \pi \int \frac{dy}{y^2} + \rho \left( \pi x^2 \right)(0.25) = 24.35 \rho$$

Hence,

$$k_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{134.03 \rho}{24.35 \rho}} = 2.35 \text{ ft}$$

Ans.

For $k_x$: $0.25 \text{ ft} < y \leq 4 \text{ ft}$

$$dI_x = \frac{1}{4} dm x^2 + dm y^2$$

$$= \frac{1}{4} \left( \rho \pi \frac{dy}{y^2} \right) \left( \frac{x}{y} \right)^2 + \left( \rho \pi \frac{dy}{y^2} \right) y^2$$

$$= \rho \pi \left( \frac{1}{y^2} + 1 \right) dy$$

$$I'_x = \int dI_x = \rho \pi \int \left( \frac{1}{y^2} + 1 \right) dy = 28.53 \rho$$

$$I''_x = \frac{1}{4} \left[ \rho \pi (4)^2 (0.25) \right] (4)^2 + \left( \rho \pi (4)^2 (0.25) \right) (0.125)^2$$

$$= 50.46 \rho$$

$$I_x = I'_x + I''_x = 28.53 \rho + 50.46 \rho = 78.99 \rho$$

Hence,

$$k_x = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{78.99 \rho}{24.35 \rho}} = 1.80 \text{ ft}$$

Ans: $k_y = 2.35 \text{ ft}$

Ans: $k_x = 1.80 \text{ ft}$
21–11.

Determine the moment of inertia of the cylinder with respect to the \(a-a\) axis of the cylinder. The cylinder has a mass \(m\).

**SOLUTION**

The mass of the differential element is \(dm = \rho dV = \rho \left( \pi a^2 \right) dy\).

\[
dI_{aa} = \frac{1}{2} \, dm \, a^2 + dm \left( y^2 \right)
\]
\[
= \frac{1}{2} \left[ \rho \left( \pi a^2 \right) dy \right] a^2 + \left[ \rho \left( \pi a^2 \right) dy \right] y^2
\]
\[
= \left( \frac{1}{2} \rho \pi a^4 + \rho \pi a^2 y^2 \right) dy
\]

\[
I_{aa} = \int_0^h \! dI_{aa} = \int_0^h \! \left( \frac{1}{2} \rho \pi a^4 + \rho \pi a^2 y^2 \right) dy
\]
\[
= \frac{\rho \pi a^2 h}{12} \left( 3a^2 + 4h^2 \right)
\]

However,

\[
m = \int_m \! dm = \int_0^h \! \rho \left( \pi a^2 \right) dy = \rho \pi a^2 h
\]

Hence,

\[
I_{aa} = \frac{m}{12} \left( 3a^2 + 4h^2 \right) \quad \text{Ans.}
\]
*21–12.

Determine the moment of inertia \( I_x \) of the composite plate assembly. The plates have a specific weight of 6 lb/ft².

**SOLUTION**

Horizontal plate:

\[
I_{xx} = \frac{1}{12} \left( \frac{6(1)(1)}{32.2} \right)(1)^2 = 0.0155
\]

Vertical plates:

\[
I_{x'y'} = \frac{1}{3} \left( \frac{6(\frac{1}{2})(1\sqrt{2})}{32.2} \right)(\frac{1}{4})^2 = 0.001372
\]

\[
I_{y'y'} = \left( \frac{6(\frac{1}{2})(1\sqrt{2})}{32.2} \right)(\frac{1}{12}) \left[ (\frac{1}{4})^2 + (1\sqrt{2})^2 \right] + \left( \frac{6(\frac{1}{2})(1\sqrt{2})}{32.2} \right)(\frac{1}{8})^2
\]

\[
= 0.01235
\]

Using Eq. 21–5,

\[
I_{xx} = (0.707)^2(0.001372) + (0.707)^2(0.01235)
\]

\[
= 0.00686
\]

Thus,

\[
I_{xx} = 0.0155 + 2(0.00686) = 0.0292 \text{ slug} \cdot \text{ft}^2
\]

**Ans:**

\[
I_{xx} = 0.0292 \text{ slug} \cdot \text{ft}^2
\]

Determine the product of inertia \( I_{yz} \) of the composite plate assembly. The plates have a weight of 6 lb/ft\(^2\).

**SOLUTION**

Due to symmetry,

\[ I_{yz} = 0 \]

Ans:

\[ I_{yz} = 0 \]
**21–14.**

Determine the products of inertia $I_{xy}$, $I_{yz}$, and $I_{xz}$ of the thin plate. The material has a density per unit area of $50 \text{ kg/m}^2$.

**SOLUTION**

The masses of segments 1 and 2 shown in Fig. a are $m_1 = 50(0.4)(0.4) = 8 \text{ kg}$ and $m_2 = 50(0.4)(0.2) = 4 \text{ kg}$. Due to symmetry $T_{x'y'} = T_{y'z'} = T_{x'z'} = 0$ for segment 1 and $T_{x'y'} = T_{y'z'} = T_{x'z'} = 0$ for segment 2.

\[
I_{xy} = \sum T_{x'y'} + mx_Gy_G
\]
\[
= \left[0 + 8(0.2)(0.2)\right] + \left[0 + 4(0)(0.2)\right]
\]
\[
= 0.32 \text{ kg} \cdot \text{m}^2
\]

\[
I_{yz} = \sum T_{y'z'} + my_Gz_G
\]
\[
= \left[0 + 8(0.2)(0)\right] + \left[0 + 4(0.2)(0.1)\right]
\]
\[
= 0.08 \text{ kg} \cdot \text{m}^2
\]

\[
I_{xz} = \sum T_{x'z'} + mx_Gz_G
\]
\[
= \left[0 + 8(0.2)(0)\right] + \left[0 + 4(0)(0.1)\right]
\]
\[
= 0
\]

**Ans:**

\[
I_{xy} = 0.32 \text{ kg} \cdot \text{m}^2
\]

\[
I_{yz} = 0.08 \text{ kg} \cdot \text{m}^2
\]

\[
I_{xz} = 0
\]
21–15.

Determine the moment of inertia of both the 1.5-kg rod and 4-kg disk about the \( z' \) axis.

**SOLUTION**

Due to symmetry \( I_{xy} = I_{yz} + I_{zx} = 0 \)

\[
I_y = I_x = \left[ \frac{1}{4} (4)(0.1)^2 + 4(0.3)^2 \right] + \frac{1}{3} (1.5)(0.3)^2 \\
= 0.415 \text{ kg} \cdot \text{m}^2
\]

\[
I_z = \frac{1}{2} (4)(0.1)^2 = 0.02 \text{ kg} \cdot \text{m}^2
\]

\[
u_x = \cos (18.43^\circ) = 0.9487, \quad u_y = \cos 90^\circ = 0,
\]

\[
u_z = \cos (90^\circ + 18.43^\circ) = -0.3162
\]

\[
I_{zz'} = I_x u_x^2 + I_y u_y^2 + I_z u_z^2 - 2I_{xy} u_x u_y - 2I_{yz} u_y u_z - 2I_{zx} u_z u_x
\]

\[
= 0.415(-0.3162)^2 + 0 + 0.02(0.9487)^2 - 0 - 0 - 0
\]

\[
= 0.0595 \text{ kg} \cdot \text{m}^2 \quad \text{Ans.}
\]
**21-16.**

The bent rod has a mass of 3 kg/m. Determine the moment of inertia of the rod about the O-a axis.

**SOLUTION**

The bent rod is subdivided into three segments and the location of center of mass for each segment is indicated in Fig. a. The mass of each segments is 

\[ m_1 = 3(1) = 3 \text{ kg}, \]

\[ m_2 = 3(0.5) = 1.5 \text{ kg}, \]

\[ m_3 = 3(0.3) = 0.9 \text{ kg}. \]

\[ I_{xx} = \frac{1}{12} (3)(1^2) + \left[ 0 + 1.5(1^2) \right] + \left[ \frac{1}{12}(0.9)(0.3^2) + 0.9(0.15^2 + 1^2) \right] \]

\[ = 3.427 \text{ kg} \cdot \text{m}^2 \]

\[ I_{yy} = \frac{1}{12} (1.5)(0.5^2) + 1.5(0.25^2) + \frac{1}{12}(0.9)(0.3^2) + 0.9(0.15^2 + 0.5^2) \]

\[ = 0.377 \text{ kg} \cdot \text{m}^2 \]

\[ I_{zz} = \frac{1}{12} (3)(1^2) + \left[ \frac{1}{12}(1.5)(0.5^2) + 1.5(1^2 + 0.25^2) \right] + \left[ 0 + 0.9(1^2 + 0.5^2) \right] \]

\[ = 3.75 \text{ kg} \cdot \text{m}^2 \]

\[ I_{xy} = [0 + 0] + [0 + 1.5(0.25)(-1)] + [0 + 0.9(0.5)(-1)] = -0.825 \text{ kg} \cdot \text{m}^2 \]

\[ I_{yx} = [0 + 0] + [0 + 0] + [0 + 0.9(-1)(0.15)] = -0.135 \text{ kg} \cdot \text{m}^2 \]

\[ I_{xz} = [0 + 0] + [0 + 0] + [0 + 0.9(0.15)(0.5)] = 0.0675 \text{ kg} \cdot \text{m}^2 \]

The unit vector that defines the direction of the O_a axis is

\[ \mathbf{U}_O = \frac{0.5i - 1j + 0.3k}{\sqrt{0.5^2 + (-1)^2 + 0.3^2}} = \frac{0.5}{\sqrt{1.34}}i - \frac{1}{\sqrt{1.34}}j + \frac{0.3}{\sqrt{1.34}}k \]

Thus, \[ u_x = \frac{0.5}{\sqrt{1.34}} \quad u_y = -\frac{1}{\sqrt{1.34}} \quad u_z = \frac{0.3}{\sqrt{1.34}} \]
*21–16. Continued

Then

\[ I_{Oa} = I_{xx} u_x^2 + I_{yy} u_y^2 + I_{zz} u_z^2 - 2I_{xy} u_x u_y - 2I_{xz} u_x u_z - 2I_{zy} u_y u_z \]

\[ = 3.427 \left( \frac{0.5}{\sqrt{1.34}} \right)^2 + 0.377 \left( -\frac{1}{\sqrt{1.34}} \right)^2 + 3.75 \left( \frac{0.3}{\sqrt{1.34}} \right)^2 - 2(-0.825) \left( \frac{0.5}{\sqrt{1.34}} \right) \left( -\frac{1}{\sqrt{1.34}} \right) \]

\[ -2(-0.135) \left( -\frac{1}{\sqrt{1.34}} \right) \left( \frac{0.3}{\sqrt{1.34}} \right) - 2(0.0675) \left( \frac{0.3}{\sqrt{1.34}} \right) \left( 0.5 \right) \]

\[ = 0.4813 \text{ kg} \cdot \text{m}^2 = 0.481 \text{ kg} \cdot \text{m}^2 \]

**Ans.**

\[ I_{Oa} = 0.481 \text{ kg} \cdot \text{m}^2 \]
21–17.
The bent rod has a weight of 1.5 lb/ft. Locate the center of gravity \( G(\bar{x}, \bar{y}) \) and determine the principal moments of inertia \( I_x, I_y, \) and \( I_z \) of the rod with respect to the \( x', y', z' \) axes.

**SOLUTION**

Due to symmetry

\[
\bar{y} = 0.5 \text{ ft}
\]

\[
\bar{x} = \frac{\sum Wx}{\sum W} = \frac{(-1)(1.5)(1) + 2(-0.5)(1.5)(1)}{3[1.5(1)]} = -0.667 \text{ ft}
\]

\[
I_x = 2 \left[ \frac{1.5}{32.2} \right] (0.5)^2 + \frac{1}{12} \left[ \frac{1.5}{32.2} \right] (1)^2 = 0.0272 \text{ slug} \cdot \text{ft}^2
\]

\[
I_y = 2 \left[ \frac{1}{12} \left( \frac{1.5}{32.2} \right) (1)^2 + \left( \frac{1.5}{32.2} \right) (0.667 - 0.5)^2 \right] + \left( \frac{1.5}{32.2} \right) (1 - 0.667)^2 = 0.0155 \text{ slug} \cdot \text{ft}^2
\]

\[
I_z = 2 \left[ \frac{1}{12} \left( \frac{1.5}{32.2} \right) (1)^2 + \left( \frac{1.5}{32.2} \right) (0.5^2 + 0.1667^2) \right] + \frac{1}{12} \left[ \frac{1.5}{32.2} \right] (1)^2 + \left( \frac{1.5}{32.2} \right) (0.3333)^2 = 0.0427 \text{ slug} \cdot \text{ft}^2
\]

**Ans:**

\[
\bar{y} = 0.5 \text{ ft}
\]

\[
\bar{x} = -0.667 \text{ ft}
\]

\[
I_x = 0.0272 \text{ slug} \cdot \text{ft}^2
\]

\[
I_y = 0.0155 \text{ slug} \cdot \text{ft}^2
\]

\[
I_z = 0.0427 \text{ slug} \cdot \text{ft}^2
\]
21–18.

Determine the moment of inertia of the rod-and-disk assembly about the $x$ axis. The disks each have a weight of 12 lb. The two rods each have a weight of 4 lb, and their ends extend to the rims of the disks.

**SOLUTION**

For a rod:

$\theta = \tan^{-1} \left( \frac{1}{1} \right) = 45^\circ$

$u_x' = \cos 90^\circ = 0, \quad u_y' = \cos 45^\circ = 0.7071, \quad u_z' = \cos (90^\circ + 45^\circ) = -0.7071$

$I_x' = I_y' = \frac{1}{12} \left( \frac{4}{32.2} \right) \left( (2)^2 + (2)^2 \right) = 0.08282 \text{ slug} \cdot \text{ft}^2$

$I_y' = 0$

$I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$

$I_x = 0 + 0 + (0.08282)(-0.7071)^2 = 0.04141 \text{ slug} \cdot \text{ft}^2$

For a disk:

$I_x = \frac{1}{2} \left( \frac{12}{32.2} \right) (1)^2 = 0.1863 \text{ slug} \cdot \text{ft}^2$

Thus.

$I_x = 2(0.04141) + 2(0.1863) = 0.455 \text{ slug} \cdot \text{ft}^2 \quad \text{Ans.}$

Ans:

$I_x = 0.455 \text{ slug} \cdot \text{ft}^2$
21–19.

Determine the moment of inertia of the composite body about the \( \alpha \alpha \) axis. The cylinder weighs 20 lb, and each hemisphere weighs 10 lb.

**SOLUTION**

\( u_{x\z} = 0.707 \)

\( u_{x\x} = 0 \)

\( u_{y\y} = 0.707 \)

\[
I_{zz} = \frac{1}{2} \left( \frac{20}{32.2} \right) (1)^2 + 2 \left[ \frac{1}{5} \left( \frac{10}{32.2} \right) (1)^2 \right]
\]

\[= 0.5590 \text{ slug} \cdot \text{ft}^2\]

\[
I_{xx} = I_{yy} = \frac{1}{12} \left( \frac{20}{32.2} \right) [3(1)^2 + (2)^2] + 2 \left[ 0.259 \left( \frac{10}{32.2} \right) (1)^2 + \frac{10}{32.2} \right]
\]

\[= 1.6975 \text{ slug} \cdot \text{ft}^2\]

\[
I_{xx} = I_{yy} = 0 + (0.707)^2 (1.6975) + (0.707)^2 (0.559)
\]

\[
I_{xx} = 1.13 \text{ slug} \cdot \text{ft}^2
\]

**Ans:**

\[
I_{xx} = 1.13 \text{ slug} \cdot \text{ft}^2
\]

Determine the moment of inertia of the disk about the axis of shaft \( AB \). The disk has a mass of 15 kg.

**SOLUTION**

Due to symmetry

\[
I_{xy} = I_{yz} = I_{zx} = 0
\]

\[
I_x = I_z = \frac{1}{4}(15)(0.15)^2 = 0.084375 \text{ kg \cdot m}^2
\]

\[
I_y = \frac{1}{2}(15)(0.15)^2 = 0.16875 \text{ kg \cdot m}^2
\]

\[
u_x = \cos 90^\circ = 0, \quad u_y = \cos 30^\circ = 0.8660
\]

\[
u_z = \cos (30^\circ + 90^\circ) = -0.5
\]

\[
I_y' = I_x u_x^2 + I_y u_y^2 + I_z u_z^2 - 2I_{xy} u_x u_y - 2I_{yz} u_y u_z - 2I_{zx} u_z u_x
\]

\[
= 0 + 0.16875(0.8660)^2 + 0.084375(-0.5)^2 - 0 - 0 - 0
\]

\[
= 0.148 \text{ kg \cdot m}^2 \quad \text{Ans.}
\]

\[
I_y' = 0.148 \text{ kg \cdot m}^2
\]
The thin plate has a weight of 5 lb and each of the four rods weighs 3 lb. Determine the moment of inertia of the assembly about the \( z \) axis.

**SOLUTION**

For the rod:

\[
I_z = \frac{1}{12} \left( \frac{3}{32.2} \right) \left( \sqrt{0.5^2 + (0.5)^2} \right)^2 = 0.003882 \text{slug} \cdot \text{ft}^2
\]

For the composite assembly of rods and disks:

\[
I_z = 4 \left[ 0.003882 + \left( \frac{3}{32.2} \right) \left( \frac{\sqrt{0.5^2 + (0.5)^2}}{2} \right)^2 \right] + \frac{1}{12} \left( \frac{5}{32.2} \right) (1^2 + 1^2)
\]

\[
= 0.0880 \text{slug} \cdot \text{ft}^2
\]

\[\text{Ans.}\]

\[I_z = 0.0880 \text{slug} \cdot \text{ft}^2\]
If a body contains no planes of symmetry, the principal moments of inertia can be determined mathematically. To show how this is done, consider the rigid body which is spinning with an angular velocity \( \omega \), directed along one of its principal axes of inertia. If the principal moment of inertia about this axis is \( I \), the angular momentum can be expressed as \( H = I \omega = I \omega_x i + I \omega_y j + I \omega_z k \). The components of \( H \) may also be expressed by Eqs. 21–10, where the inertia tensor is assumed to be known. Equate the \( i \), \( j \), and \( k \) components of both expressions for \( H \) and consider \( \omega_x \), \( \omega_y \), and \( \omega_z \) to be unknown. The solution of these three equations is obtained provided the determinant of the coefficients is zero. Show that this determinant, when expanded, yields the cubic equation

\[
I^3 - (I_{xx} + I_{yy} + I_{zz}) I^2 + (I_{xx} I_{yy} + I_{yy} I_{zz} + I_{zz} I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2) I \\
- (I_{xx} I_{yy} I_{zz} - 2I_{xy} I_{yz} I_{zx} - I_{xx} I_{zz}^2 - I_{yy} I_{xx}^2 - I_{zz} I_{yy}^2) = 0
\]

The three positive roots of \( I \), obtained from the solution of this equation, represent the principal moments of inertia \( I_x \), \( I_y \), and \( I_z \).

**SOLUTION**

\[
H = I \omega = I \omega_x i + I \omega_y j + I \omega_z k
\]

Equating the \( i \), \( j \), \( k \) components to the scalar equations (Eq. 21–10) yields

\[
(I_{xx} - I) \omega_x - I_{xy} \omega_y - I_{xz} \omega_z = 0
\]

\[
-I_{xx} \omega_x + (I_{xy} - I) \omega_y - I_{yz} \omega_z = 0
\]

\[
-I_{xz} \omega_z - I_{zy} \omega_y + (I_{zz} - I) \omega_z = 0
\]

Solution for \( \omega_x \), \( \omega_y \), and \( \omega_z \) requires

\[
\begin{vmatrix}
(I_{xx} - I) & -I_{xy} & -I_{xz} \\
-I_{xy} & (I_{yy} - I) & -I_{yz} \\
-I_{xz} & -I_{yz} & (I_{zz} - I)
\end{vmatrix} = 0
\]

Expanding

\[
I^3 - (I_{xx} + I_{yy} + I_{zz}) I^2 + (I_{xx} I_{yy} + I_{yy} I_{zz} + I_{zz} I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2) I \\
- (I_{xx} I_{yy} I_{zz} - 2I_{xy} I_{yz} I_{zx} - I_{xx} I_{zz}^2 - I_{yy} I_{xx}^2 - I_{zz} I_{yy}^2) = 0 \text{ Q.E.D.}
\]

**Ans:**

\[
I^3 - (I_{xx} + I_{yy} + I_{zz}) I^2 + (I_{xx} I_{yy} + I_{yy} I_{zz} + I_{zz} I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2) I \\
- (I_{xx} I_{yy} I_{zz} - 2I_{xy} I_{yz} I_{zx} - I_{xx} I_{zz}^2 - I_{yy} I_{xx}^2 - I_{zz} I_{yy}^2) = 0 \text{ Q.E.D.}
\]
21–23.

Show that if the angular momentum of a body is determined with respect to an arbitrary point A, then $\mathbf{H}_A$ can be expressed by Eq. 21–9. This requires substituting $\rho_A = \rho_G + \rho_{G/A}$ into Eq. 21–6 and expanding, noting that $\int \rho_G dm = 0$ by definition of the mass center and $v_G = v_A + \omega \times \rho_{G/A}$.

**SOLUTION**

$$
\mathbf{H}_A = \left( \int_m \rho_A dm \right) \times v_A + \int_m \rho_A \times (\omega \times \rho_A) dm
$$

$$
= \left( \int_m (\rho_G + \rho_{G/A}) dm \right) \times v_A + \left( \int_m \rho_G dm \right) \times (\rho_{G/A} \times v_A) + \int_m (\rho_G \times (\rho_{G/A} \times v_A) + \rho_{G/A} \times (\omega \times \rho_G) dm
$$

Since $\int_m \rho_G dm = 0$ and from Eq. 21–8 $\mathbf{H}_G = \int_m \rho_G \times (\omega \times \rho_G) dm$

$$
\mathbf{H}_A = (\rho_{G/A} \times v_A) m + \mathbf{H}_G + \rho_{G/A} \times (\omega \times \rho_{G/A}) m
$$

$$
= \rho_{G/A} \times (v_A + (\omega \times \rho_{G/A})) m + \mathbf{H}_G
$$

$$
= (\rho_{G/A} \times m v_G) + \mathbf{H}_G
$$

Q.E.D.
*21–24.

The 15-kg circular disk spins about its axe with a constant angular velocity of $\omega_1 = 10 \text{ rad/s}$. Simultaneously, the yoke is rotating with a constant angular velocity of $\omega_2 = 5 \text{ rad/s}$. Determine the angular momentum of the disk about its center of mass $O$, and its kinetic energy.

**SOLUTION**

The mass moments of inertia of the disk about the $x$, $y$, and $z$ axes are

$$I_x = I_z = \frac{1}{4} mr^2 = \frac{1}{4} (15)(0.15^2) = 0.084375 \text{ kg} \cdot \text{m}^2$$

$$I_y = \frac{1}{2} mr^2 = \frac{1}{2} (15)(0.15^2) = 0.16875 \text{ kg} \cdot \text{m}^2$$

Due to symmetry,

$I_{xy} = I_{yz} = I_{xz} = 0$

Here, the angular velocity of the disk can be determined from the vector addition of $\omega_1$ and $\omega_2$. Thus,

$$\omega = \omega_1 + \omega_2 = [-10\hat{j} + 5\hat{k}] \text{ rad/s}$$

so that

$$\omega_x = 0 \quad \omega_y = -10 \text{ rad/s} \quad \omega_z = 5 \text{ rad/s}$$

Since the disk rotates about a fixed point $O$, we can apply

$$H_x = I_x \omega_x = 0.084375(0) = 0$$

$$H_y = I_y \omega_y = 0.16875(-10) = -1.6875 \text{ kg} \cdot \text{m}^2/\text{s}$$

$$H_z = I_z \omega_z = 0.084375(5) = 0.421875 \text{ kg} \cdot \text{m}^2/\text{s}$$

Thus,

$$H_O = [-1.69\hat{j} + 0.422\hat{k}] \text{ kg} \cdot \text{m}^2/\text{s} \quad \text{Ans.}$$

The kinetic energy of the disk can be determined from

$$T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2$$

$$= \frac{1}{2}(0.084375)(0^2) + \frac{1}{2}(0.16875)(-10)^2 + \frac{1}{2}(0.084375)(5^2)$$

$$= 9.49 \text{ J} \quad \text{Ans.}$$
21–25.

The large gear has a mass of 5 kg and a radius of gyration of $k_z = 75$ mm. Gears $B$ and $C$ each have a mass of 200 g and a radius of gyration about the axis of their connecting shaft of 15 mm. If the gears are in mesh and $C$ has an angular velocity of $\omega_C = [15j]$ rad/s, determine the total angular momentum for the system of three gears about point $A$.

**SOLUTION**

\[
I_A = 5(0.075)^2 = 28.125 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\
I_B = I_C = 0.2(0.015)^2 = 45 \times 10^{-6} \text{ kg} \cdot \text{m}^2
\]

Kinematics:

\[
\omega_C = \omega_B = 15 \text{ rad/s} \\
v = (0.04)(15) = 0.6 \text{ m/s} \\
\omega_A = \left(\frac{0.6}{0.1}\right) = 6 \text{ rad/s}
\]

\[
H_B = I_B \omega_B = (45 \times 10^{-6})(15) = 675 \times 10^{-6}
\]

\[
H_B = -675 \times 10^{-6} \sin 45^\circ \mathbf{i} - 675 \times 10^{-6} \cos 45^\circ \mathbf{j}
\]

\[
H_B = -477.3 \times 10^{-6} \mathbf{i} - 477.3 \times 10^{-6} \mathbf{j}
\]

\[
H_C = I_C \omega_C = (45 \times 10^{-6})(15) = 675 \times 10^{-6}
\]

\[
H_C = 675 \times 10^{-6} \mathbf{j}
\]

\[
H_A = I_A \omega_A = 28.125 \times 10^{-3}(6) = 0.16875
\]

\[
H_A = 0.16875 \mathbf{k}
\]

The total angular momentum is therefore,

\[
H = H_B + H_C + H_A = \{-477(10^{-6}) \mathbf{i} + 198(10^{-6}) \mathbf{j} + 0.169 \mathbf{k}\} \text{ kg} \cdot \text{m}^2/\text{s} \quad \text{Ans.}
\]

**Ans:**

\[
H = \{-477(10^{-6}) \mathbf{i} + 198(10^{-6}) \mathbf{j} + 0.169 \mathbf{k}\} \text{ kg} \cdot \text{m}^2/\text{s}
\]

The circular disk has a weight of 15 lb and is mounted on the shaft \( AB \) at an angle of 45° with the horizontal. Determine the angular velocity of the shaft when a constant torque \( M = 2 \text{ lb} \cdot \text{ft} \) is applied to the shaft. The shaft is originally spinning at \( \omega_1 = 8 \text{ rad/s} \) when the torque is applied.

**SOLUTION**

Due to symmetry

\[ I_{xy} = I_{yz} = I_{xz} = 0 \]

\[ I_x = I_x = \frac{1}{2}(\frac{15}{322}) (0.8)^2 = 0.07453 \text{ slug} \cdot \text{ft}^2 \]

\[ I_z = \frac{1}{2}(\frac{15}{322}) (0.8)^2 = 0.1491 \text{ slug} \cdot \text{ft}^2 \]

For \( x' \) axis

\[ u_x = \cos 45° = 0.7071 \quad u_y = \cos 45° = 0.7071 \]

\[ u_z = \cos 90° = 0 \]

\[ I_{x'} = I_x u_x^2 + I_y u_y^2 + I_z u_z^2 - 2I_{xy} u_x u_y - 2I_{yz} u_y u_z - 2I_{xz} u_z u_x \]

\[ = 0.1491(0.7071)^2 + 0.07453(0.7071)^2 + 0 - 0 - 0 = 0.1118 \text{ slug} \cdot \text{ft}^2 \]

Principle of impulse and momentum:

\[ (H_{x'})_1 + \sum \int M_{x'} \, dt = (H_{x'})_2 \]

\[ 0.1118(8) + 2(3) = 0.1118 \omega_2 \]

\[ \omega_2 = 61.7 \text{ rad/s} \quad \text{Ans.} \]
21–27.

The circular disk has a weight of 15 lb and is mounted on the shaft $AB$ at an angle of $45^\circ$ with the horizontal. Determine the angular velocity of the shaft when a torque $M = (4e^{0.1t})$ lb·ft, where $t$ is in seconds, is applied to the shaft. The shaft is originally spinning at $\omega_1 = 8$ rad/s when the torque is applied.

**SOLUTION**

Due to symmetry

$I_{xy} = I_{yz} = I_{zx} = 0$

$I_x = I_z = \frac{1}{2} \left( \frac{15}{32} \right) (0.8)^2 = 0.07453 \text{ slug} \cdot \text{ft}^2$

$I_x = \frac{1}{2} \left( \frac{15}{32} \right) (0.8)^2 = 0.1491 \text{ slug} \cdot \text{ft}^2$

For $x'$ axis

$u_x = \cos 45^\circ = 0.7071  \quad u_y = \cos 45^\circ = 0.7071$

$u_z = \cos 90^\circ = 0$

$I_x = I_x u_x^2 + I_y u_y^2 + I_z u_z^2 - 2I_{xy} u_x u_y - 2I_{yx} u_y u_x - 2I_{zx} u_z u_x$

$= 0.1491(0.7071)^2 + 0.07453(0.7071)^2 + 0 - 0 - 0 - 0$

$= 0.1118 \text{ slug} \cdot \text{ft}^2$

Principle of impulse and momentum:

$(H_x)_1 + \sum \int M_x \, dt = (H_x)_2$

$0.1118(8) + \int_0^2 4e^{0.1t} \, dt = 0.1118\omega_2$

$\omega_2 = 87.2 \text{ rad/s}$

**Ans:**

$\omega_2 = 87.2 \text{ rad/s}$
The rod assembly is supported at $G$ by a ball-and-socket joint. Each segment has a mass of 0.5 kg. If the assembly is originally at rest and an impulse of $I = (-8k)$ N·s is applied at $D$, determine the angular velocity of the assembly just after the impact.

**SOLUTION**

Moments and products of inertia:

\[ I_{xx} = \frac{1}{12} [2(0.5)](2)^2 + 2(0.5(0.5))(1)^2 = 0.8333 \text{ kg} \cdot \text{m}^2 \]

\[ I_{yy} = \frac{1}{12} [1(0.5)](1)^2 = 0.04166 \text{ kg} \cdot \text{m}^2 \]

\[ I_{zz} = \frac{1}{12} [2(0.5)](2)^2 + 2\left(\frac{1}{12}[0.5(0.5)](0.5)^2 + 0.5(0.5)(1^2 + 0.25^2)\right) = 0.875 \text{ kg} \cdot \text{m}^2 \]

\[ I_{xy} = \left[0.5(0.5)\right][-0.25] + \left[0.5(0.5)\right][0.25](1) = -0.125 \text{ kg} \cdot \text{m}^2 \]

\[ I_{yz} = I_{xz} = 0 \]

From Eq. 21–10

\[ H_x = 0.8333\omega_x + 0.125\omega_y \]

\[ H_y = 0.125\omega_x + 0.04166\omega_y \]

\[ H_z = 0.875\omega_z \]

\[ (H_G)_1 + \int_{t_i}^{t_f} M_G dt = (H_G)_2 \]

\[ \mathbf{0} + (-0.5\mathbf{i} + \mathbf{j}) \times (-8\mathbf{k}) = (0.8333\omega_x + 0.125\omega_y)\mathbf{i} + (0.125\omega_x + 0.04166\omega_y)\mathbf{j} + 0.875\omega_z\mathbf{k} \]

Equating $i$, $j$ and $k$ components

\[-8 = 0.8333\omega_x + 0.125\omega_y \quad (1) \]

\[-4 = 0.125\omega_x + 0.04166\omega_y \quad (2) \]

\[0 = 0.875\omega_z \quad (3) \]

Solving Eqs. (1) to (3) yields:

\[ \omega_x = 8.73 \text{ rad/s} \quad \omega_y = -122 \text{ rad/s} \quad \omega_z = 0 \]

Then \[ \omega = [8.73\mathbf{i} - 122\mathbf{j}] \text{ rad/s} \quad \text{Ans.} \]

\[ \text{Ans:} \]

\[ \omega = [8.73\mathbf{i} - 122\mathbf{j}] \text{ rad/s} \]
21–29.

The 4-lb rod $AB$ is attached to the 1-lb collar at $A$ and a 2-lb link $BC$ using ball-and-socket joints. If the rod is released from rest in the position shown, determine the angular velocity of the link after it has rotated 180°.

**SOLUTION**

$T_1 + V_1 = T_2 + V_2$

$0 + 4(0.25) + 2(0.25) = T_2 - 4(0.25) - 2(0.25)$

$T_2 = 3$

$\omega_{AB} = \frac{0.5\omega_x}{1.3} = 0.3846\omega_x$

$T_2 = \frac{1}{2}(\frac{2}{3})^2(0.5)^2\omega_x^2 + \frac{1}{2}(\frac{4}{32.2})^2(1.3)^2(0.3846\omega_x)^2 + \frac{1}{2}(\frac{4}{32.2})^2(0.25\omega_x)^2$

$T_2 = 0.007764\omega_x^2 = 3$

$\omega_x = 19.7\text{ rad/s}$

**Ans:**

$\omega_x = 19.7\text{ rad/s}$
21–30.

The rod weighs 3 lb/ft and is suspended from parallel cords at A and B. If the rod has an angular velocity of 2 rad/s about the z axis at the instant shown, determine how high the center of the rod rises at the instant the rod momentarily stops swinging.

SOLUTION

\[ T_1 + V_1 = T_2 + V_2 \]

\[ \frac{1}{2} \left( \frac{1}{12} \cdot \frac{W}{g} \right) l^2 \omega^2 + 0 = 0 + Wh \]

\[ h = \frac{1}{24} \cdot \frac{l^2 \omega^2}{g} = \frac{1}{24} \cdot \frac{(6)^2(2)^2}{(32.2)} \]

\[ h = 0.1863 \text{ ft} = 2.24 \text{ in.} \]

Ans.

\[ h = 2.24 \text{ in.} \]
The 4-lb rod $AB$ is attached to the rod $BC$ and collar $A$ using ball-and-socket joints. If $BC$ has a constant angular velocity of $2 \text{ rad/s}$, determine the kinetic energy of $AB$ when it is in the position shown. Assume the angular velocity of $AB$ is directed perpendicular to the axis of $AB$.

**SOLUTION**

\[
v_A = v_B + [-2\text{j}] \text{ ft/s} \quad \omega_{AB} = \omega_x \text{i} + \omega_y \text{j} + \omega_z \text{k}
\]

\[
r_{B/A} = [-3\text{i} + 1\text{j} + 1\text{k}] \text{ ft} \quad r_{G/B} = [1.5\text{i} - 0.5\text{j} - 0.5\text{k}]
\]

\[
v_B = v_A + \omega_{AB} \times r_{B/A}
\]

\[
-2\text{j} = v_A \text{i} + \begin{vmatrix} 1 & \text{j} & \text{k} \\ \omega_x & \omega_y & \omega_z \\ -3 & 1 & 1 \end{vmatrix}
\]

Equating $\text{j}$, $\text{k}$ components

\[
\omega_y - \omega_z + v_A = 0 \quad (1)
\]

\[
\omega_x + 3\omega_z = 2 \quad (2)
\]

\[
\omega_x + 3\omega_y = 0 \quad (3)
\]

Since $\omega_{AB}$ is perpendicular to the axis of the rod,

\[
\omega_{AB} \quad r_{B/A} = (\omega_x \text{i} + \omega_y \text{j} + \omega_z \text{k}) \cdot (-3\text{i} + 1\text{j} + 1\text{k}) = 0
\]

\[
-3\omega_x + 1\omega_y + 1\omega_z = 0 \quad (4)
\]

Solving Eqs. (1) to (4) yields:

\[
\omega_x = 0.1818 \text{ rad/s} \quad \omega_y = -0.06061 \text{ rad/s} \quad \omega_z = 0.6061 \text{ rad/s}
\]

\[
v_A = 0.6667 \text{ ft/s}
\]

Hence $\omega_{AB} = [0.1818\text{i} - 0.06061\text{j} + 0.6061\text{k}] \text{ rad/s}$, $v_A = [0.6667\text{i}] \text{ ft/s}$

\[
v_G = v_B + \omega_{AB} \times r_{G/B}
\]

\[
= -2\text{j} + \begin{vmatrix} 1 & \text{j} & \text{k} \\ 0.1818 & -0.06061 & 0.6061 \\ 1.5 & -0.5 & -0.5 \end{vmatrix}
\]

\[
= [0.3333\text{i} - 1.0\text{j}] \text{ ft/s}
\]

\[
\omega_{AB}^2 = 0.1818^2 + (-0.06061)^2 + 0.6061^2 = 0.4040
\]

\[
v_G^2 = (0.3333)^2 + (-1.0)^2 = 1.111
\]

\[
T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G \omega_{AB}^2
\]

\[
= \frac{1}{2} \left( \frac{4}{32.2} \right) (1.111) + \frac{1}{2} \left( \frac{4}{32.2} \right) \left( \sqrt{3^2 + 1^2 + 1^2} \right)^2 (0.4040)
\]

\[
= 0.0920 \text{ ft} \cdot \text{lb}
\]

\[\text{Ans.}\]

\[T = 0.0920 \text{ ft} \cdot \text{lb}\]
The 2-kg thin disk is connected to the slender rod which is fixed to the ball-and-socket joint at A. If it is released from rest in the position shown, determine the spin of the disk about the rod when the disk reaches its lowest position. Neglect the mass of the rod. The disk rolls without slipping.

**SOLUTION**

\[
I_x = I_z = \frac{1}{4}(2)(0.1)^2 + 2(0.5)^2 = 0.505 \text{ kg} \cdot \text{m}^2
\]

\[
I_y = \frac{1}{2}(2)(0.1)^2 = 0.01 \text{ kg} \cdot \text{m}^2
\]

\[
\omega = \omega_x + \omega_z = -\omega_y \hat{j} + \omega_z \cdot \sin 11.31^\circ \hat{j} + \omega_z \cdot \cos 11.31^\circ \hat{k}
\]

\[
= (0.19612\omega_x - \omega_y)\hat{j} + (0.98058\omega_z)\hat{k}
\]

Since \( \mathbf{v}_A = \mathbf{v}_C = 0 \), then

\[
\mathbf{v}_C = \mathbf{v}_A + \omega \times \mathbf{r}_{C/A}
\]

\[
0 = 0 + [(0.19612\omega_z - \omega_y)\hat{j} + (0.98058\omega_z)\hat{k}] \times (0.5\hat{j} - 0.1\hat{k})
\]

\[
0 = -0.019612\omega_z + 0.1\omega_y - 0.49029\omega_z
\]

\[
\omega_z = 0.19612\omega_y
\]

Thus,

\[
\omega = -0.96154\omega_y \hat{j} + 0.19231\omega_y \hat{k}
\]

\[
h_1 = 0.5 \sin 41.31^\circ = 0.3301 \text{ m}, \quad h_2 = 0.5 \sin 18.69^\circ = 0.1602 \text{ m}
\]

\[
T_1 + V_1 = T_2 + V_2
\]

\[
0 + 2(9.81)(0.3301) = \left[ 0 + \frac{1}{2}(0.01)(-0.96154\omega_y)^2 + \frac{1}{2}(0.505)(0.19231\omega_y)^2 \right] - 2(9.81)(0.1602)
\]

\[
\omega_y = 26.2 \text{ rad/s}
\]

**Ans:**

\[
\omega_y = 26.2 \text{ rad/s}
\]
21–33.

The 20-kg sphere rotates about the axle with a constant angular velocity of \( \omega_s = 60 \text{ rad/s} \). If shaft \( AB \) is subjected to a torque of \( T = 50 \text{ N} \cdot \text{m} \), causing it to rotate, determine the value of \( \omega_s \) after the shaft has turned 90° from the position shown. Initially, \( \omega_p = 0 \). Neglect the mass of arm \( CDE \).

**SOLUTION**

The mass moments of inertia of the sphere about the \( x' \), \( y' \), and \( z' \) axes are

\[
I_{x'} = I_{y'} = I_{z'} = \frac{2}{5} m r^2 = \frac{2}{5} (20)(0.1^2) = 0.08 \text{ kg} \cdot \text{m}^2
\]

When the sphere is at position (1), Fig. a, \( \omega_p = 0 \). Thus, the velocity of its mass center is zero and its angular velocity is \( \omega_1 = [60\mathbf{k}] \text{ rad/s} \). Thus, its kinetic energy at this position is

\[
T = \frac{1}{2} m (v_1)^2 + \frac{1}{2} I_{x'} (\omega_1)^2 + \frac{1}{2} I_{y'} (\omega_1)^2 + \frac{1}{2} I_{z'} (\omega_1)^2
\]

\[
= 0 + 0 + 0 + \frac{1}{2} (0.08) (60^2)
\]

\[
= 144 \text{ J}
\]

When the sphere is at position (2), Fig. a, \( \omega_p = \omega_p \mathbf{j} \). Then the velocity of its mass center is \( (v_2) = (v_1) \times (v_{G/C}) = (\omega_p \mathbf{j}) \times (-0.3 \mathbf{j} + 0.4 \mathbf{k}) = -0.4 \omega_p \mathbf{j} - 0.3 \omega_p \mathbf{k} \). Then \( (v_2)^2 = (-0.4 \omega_p)^2 + (-0.3 \omega_p)^2 = 0.25 \omega_p^2 \). Also, its angular velocity at this position is \( \omega_2 = \omega_p \mathbf{j} - 60 \mathbf{j} \). Thus, its kinetic energy at this position is

\[
T = \frac{1}{2} m (v_2)^2 + \frac{1}{2} I_{x'} (\omega_2)^2 + \frac{1}{2} I_{y'} (\omega_2)^2 + \frac{1}{2} I_{z'} (\omega_2)^2
\]

\[
= \frac{1}{2} (20)(0.25 \omega_p^2) + \frac{1}{2} (0.08)(\omega_p^2) + \frac{1}{2} (0.08)(-60^2)
\]

\[
= 2.54 \omega_p^2 + 144
\]

When the sphere moves from position (1) to position (2), its center of gravity raises vertically \( \Delta z = 0.1 \text{ m} \). Thus, its weight \( W \) does negative work.

\[
U_W = -W \Delta z = -20(9.81)(0.1) = -19.62 \text{ J}
\]

Here, the couple moment \( M \) does positive work.

\[
U_W = M \theta = 50 \left( \frac{\pi}{2} \right) = 25\pi \text{J}
\]

Applying the principle of work and energy,

\[
T_1 + \Sigma U_{1-2} = T_2
\]

\[
144 + 25\pi + (-19.62) = 2.54 \omega_p^2 + 144
\]

\[
\omega_p = 4.82 \text{ rad/s}
\]

**Ans:**

\[
\omega_p = 4.82 \text{ rad/s}
\]
21–34.

The 200-kg satellite has its center of mass at point $G$. Its radii of gyration about the $z'$, $x'$, $y'$ axes are $k_x = 300$ mm, $k_y = 500$ mm, respectively. At the instant shown, the satellite rotates about the $x'$, $y'$, and $z'$ axes with the angular velocity shown, and its center of mass $G$ has a velocity of $v_G = \{-250i + 200j + 120k\}$ m/s. Determine the angular momentum of the satellite about point $A$ at this instant.

**SOLUTION**

The mass moments of inertia of the satellite about the $x', y', z'$ axes are

$$
I_x = I_y = 200(0.5^2) = 50 \text{ kg} \cdot \text{m}^2
$$

$$
I_z = 200(0.3^2) = 18 \text{ kg} \cdot \text{m}^2
$$

Due to symmetry, the products of inertia of the satellite with respect to the $x', y', z'$ coordinate system are equal to zero.

$$
I_{x'y'} = I_{y'z'} = I_{x'z'} = 0
$$

The angular velocity of the satellite is

$$
\omega = [600i + 300j + 1250k] \text{ rad/s}
$$

Thus,

$$
\omega_x = 600 \text{ rad/s} \quad \omega_y = -300 \text{ rad/s} \quad \omega_z = 1250 \text{ rad/s}
$$

Then, the components of the angular momentum of the satellite about its mass center $G$ are

$$
(H_G)_x = I_x \omega_x = 50(600) = 30000 \text{ kg} \cdot \text{m}^2/\text{s}
$$

$$
(H_G)_y = I_y \omega_y = 50(-300) = -15000 \text{ kg} \cdot \text{m}^2/\text{s}
$$

$$
(H_G)_z = I_z \omega_z = 18(1250) = 22500 \text{ kg} \cdot \text{m}^2/\text{s}
$$

Thus,

$$
H_G = [30,000i - 15,000j + 22,500k] \text{ kg} \cdot \text{m}^2/\text{s}
$$

The angular momentum of the satellite about point $A$ can be determined from

$$
H_A = r_{G/A} \times m v_G + H_G
$$

$$
= (0.8k) \times 200(-250i + 200j + 120k) + (30,000i - 15,000j + 22,500k)
$$

$$
= [-2000i - 55,000j + 22,500k] \text{ kg} \cdot \text{m}^2/\text{s}
$$

**Ans:**

$$
H_A = [-2000i - 55,000j + 22,500k] \text{ kg} \cdot \text{m}^2/\text{s}
$$
The 200-kg satellite has its center of mass at point $G$. Its radii of gyration about the $z'$, $x'$, $y'$ axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the $x'$, $y'$, and $z'$ axes with the angular velocity shown, and its center of mass $G$ has a velocity of $v_G = \{-250\hat{i} + 200\hat{j} + 120\hat{k}\}$ m/s. Determine the kinetic energy of the satellite at this instant.

**SOLUTION**

The mass moments of inertia of the satellite about the $x'$, $y'$, and $z'$ axes are

\[
I_{x'} = I_{y'} = 200(0.5^2) = 50 \text{ kg} \cdot \text{m}^2
\]

\[
I_{z'} = 200(0.3^2) = 18 \text{ kg} \cdot \text{m}^2
\]

Due to symmetry, the products of inertia of the satellite with respect to the $x'$, $y'$, and $z'$ coordinate system are equal to zero.

\[
I_{x'y'} = I_{y'z'} = I_{z'x'} = 0
\]

The angular velocity of the satellite is

\[
\mathbf{\omega} = [600\hat{i} - 300\hat{j} + 1250\hat{k}] \text{ rad/s}
\]

Thus,

\[
\omega_{x'} = 600 \text{ rad/s} \quad \omega_{y'} = -300 \text{ rad/s} \quad \omega_{z'} = 1250 \text{ rad/s}
\]

Since $v_G^2 = (-250)^2 + 200^2 + 120^2 = 116\,900 \text{ m}^2/\text{s}^2$, the kinetic energy of the satellite can be determined from

\[
T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_{x'}\omega_{x'}^2 + \frac{1}{2} I_{y'}\omega_{y'}^2 + \frac{1}{2} I_{z'}\omega_{z'}^2
\]

\[
= \frac{1}{2} (200)(116\,900) + \frac{1}{2} (50)(600^2) + \frac{1}{2} (50)(-300)^2 + \frac{1}{2} (18)(1250^2)
\]

\[
= 37.0025\times10^6\times J = 37.0 \text{ MJ}
\]

**Ans:**

\[T = 37.0 \text{ MJ}\]
**21–36.**

The 15-kg rectangular plate is free to rotate about the \( y \) axis because of the bearing supports at \( A \) and \( B \). When the plate is balanced in the vertical plane, a 3-g bullet is fired into it, perpendicular to its surface, with a velocity \( \mathbf{v} = \{-2000 \mathbf{i} \} \) m/s. Compute the angular velocity of the plate at the instant it has rotated 180°. If the bullet strikes corner \( D \) with the same velocity \( \mathbf{v} \), instead of at \( C \), does the angular velocity remain the same? Why or why not?

**SOLUTION**

Consider the projectile and plate as an entire system.

Angular momentum is conserved about the \( AB \) axis.

\[
(H_{AB})_1 = - (0.003)(2000)(0.15) \mathbf{j} = \{-0.9 \mathbf{j}\}
\]

\[
(H_{AB})_2 = (H_{AB})_2
\]

\[-0.9 \mathbf{j} = I_x \omega_x \mathbf{i} + I_y \omega_y \mathbf{j} + I_z \omega_z \mathbf{k}
\]

Equating components,

\[\omega_x = 0\]

\[\omega_z = 0\]

\[\omega_y = \frac{-0.9}{\frac{1}{12}(15)(0.15)^2 + 15(0.075)^2} = -8 \text{ rad/s}\]

\[T_1 + V_1 = T_2 + V_2\]

\[\frac{1}{2} \left[ \frac{1}{12}(15)(0.15)^2 + 15(0.075)^2 \right] (8)^2 + 15(9.81)(0.15)\]

\[= \frac{1}{2} \left[ \frac{1}{12}(15)(0.15)^2 + 15(0.075)^2 \right] \omega_{AB}^2\]

\[\omega_{AB} = 21.4 \text{ rad/s} \quad \text{Ans.}\]

If the projectile strikes the plate at \( D \), the angular velocity is the same, only the impulsive reactions at the bearing supports \( A \) and \( B \) will be different.
21–37.

The 5-kg thin plate is suspended at $O$ using a ball-and-socket joint. It is rotating with a constant angular velocity $\omega = \{2k\}$ rad/s when the corner $A$ strikes the hook at $S$, which provides a permanent connection. Determine the angular velocity of the plate immediately after impact.

**SOLUTION**

Angular momentum is conserved about the $OA$ axis.

$$(H_O)_1 = I_O \omega \cdot k$$

$$= \left[ \frac{1}{12} (5)(0.6)^2 \right] (2) k = 0.30 k$$

$$u_{OA} = \{0.6\hat{j} - 0.8k\}$$

$$(H_{OA})_1 = (H_O)_1 \cdot u_{OA}$$

$$= (0.30)(-0.8) = -0.24$$

$$\omega = 0.60\hat{j} - 0.80\hat{k}$$

$$(H_O)_2 = I_O \omega \cdot \hat{j} + I_O \omega \cdot \hat{k}$$

$$= \frac{1}{3} (5)(0.4)^2 \omega \cdot \hat{j} + \frac{1}{12} (5)(0.6)^2 \omega \cdot \hat{k}$$

$$= 0.2667 \omega \cdot \hat{j} + 0.150 \omega \cdot \hat{k}$$

From Eq. (1),

$$\omega_x = 0.6 \omega$$

$$\omega_z = -0.8 \omega$$

$$(H_O)_2 = 0.160 \omega \cdot \hat{j} - 0.120 \omega \cdot \hat{k}$$

$$(H_{OA})_2 = (H_O)_2 - u_{OA}$$

$$= 0.160 \omega (0.6) + (0.12 \omega)(0.8) = 0.192 \omega$$

Thus,

$$(H_{OA})_1 = (H_{OA})_2$$

$$-0.24 = 0.192 \omega$$

$$\omega = -1.25 \text{ rad/s}$$

$$\omega = -1.25 u_{OA}$$

$$\omega = \{-0.750 \hat{j} + 1.00 \hat{k}\} \text{ rad/s}$$

**Ans:**

$$\omega = \{-0.750 \hat{j} + 1.00 \hat{k}\} \text{ rad/s}$$
21–38.

Determine the kinetic energy of the 7-kg disk and 1.5-kg rod when the assembly is rotating about the z axis at \( \omega = 5 \text{ rad/s} \).

**SOLUTION**

Due to symmetry

\( I_{xy} = I_{yx} = I_{zx} = 0 \)

\[ I_y = I_z = \left[ \frac{1}{4}(7)(0.1)^2 + 7(0.2)^2 \right] + \frac{1}{5}(1.5)(0.2)^2 \]

\[ = 0.3175 \text{ kg} \cdot \text{m}^2 \]

\[ I_z = \frac{1}{2}(7)(0.1)^2 = 0.035 \text{ kg} \cdot \text{m}^2 \]

For z axis

\[ u_x = \cos 26.56^\circ = 0.8944 \quad u_y = \cos 116.57^\circ = -0.4472 \]

\[ u_z = \cos 90^\circ = 0 \]

\[ I_c = I_x u_x^2 + I_y u_y^2 + I_z u_z^2 - 2I_{xy}u_xu_y - 2I_{xz}u_xu_z - 2I_{yz}u_yu_z \]

\[ = 0 + 0.3175(-0.4472)^2 + 0.035(0.8944)^2 - 0 - 0 - 0 \]

\[ = 0.0915 \text{ kg} \cdot \text{m}^2 \]

\[ T = \frac{1}{2}I_c \omega_z^2 + \frac{1}{2}I_y \omega_y^2 + \frac{1}{2}I_z \omega_z^2 \]

\[ = 0 + 0 + \frac{1}{2}(0.0915)(5)^2 \]

\[ = 1.14 \text{ J} \]

Ans.
Determine the angular momentum $H_z$ of the 7-kg disk and 1.5-kg rod when the assembly is rotating about the $z$ axis at $\omega = 5 \text{ rad/s}$.

**SOLUTION**

Due to symmetry $I_{xy} = I_{yz} = I_{xz} = 0$

\[
I_y = I_x = \frac{1}{4} (7)(0.1)^2 + 7(0.2)^2 + \frac{1}{3} (1.5)(0.2)^2
\]

\[= 0.3175 \text{ kg} \cdot \text{m}^2\]

\[I_z = \frac{1}{2} (7)(0.1)^2 = 0.035 \text{ kg} \cdot \text{m}^2\]

For $z'$ axis

\[
u_z = \cos 26.56^\circ = 0.8944 \quad \nu_y = \cos 116.57^\circ = -0.4472 \]

\[\nu_x = \cos 90^\circ = 0\]

\[I_z = I_x \nu_x^2 + I_y \nu_y^2 + I_z \nu_z^2 - 2I_{xy} \nu_x \nu_y - 2I_{yz} \nu_y \nu_z - 2I_{zx} \nu_z \nu_x\]

\[= 0 + 0.3175(-0.4472)^2 + 0.035(0.8944)^2 - 0 - 0 - 0\]

\[= 0.0915 \text{ kg} \cdot \text{m}^2\]

\[\omega_x = \omega_y = 0\]

\[H_z = -I_{zx} \omega_x - I_{xy} \omega_y + I_{yz} \omega_z\]

\[= -0 - 0 + 0.0915(5)\]

\[= 0.4575 \text{ kg} \cdot \text{m}^2/\text{s}\]

**Ans:**

$H_z = 0.4575 \text{ kg} \cdot \text{m}^2/\text{s}$
*21–40.

Derive the scalar form of the rotational equation of motion about the x axis if $\Omega \neq \omega$ and the moments and products of inertia of the body are not constant with respect to time.

**SOLUTION**

In general

$$\mathbf{M} = \frac{d}{dt}(\mathbf{H}_x \mathbf{i} + \mathbf{H}_y \mathbf{j} + \mathbf{H}_z \mathbf{k})$$

$$= (\dot{\mathbf{H}}_x \mathbf{i} + \dot{\mathbf{H}}_y \mathbf{j} + \dot{\mathbf{H}}_z \mathbf{k})_{xyz} + \Omega \times (\mathbf{H}_x \mathbf{i} + \mathbf{H}_y \mathbf{j} + \mathbf{H}_z \mathbf{k})$$

Substitute $\Omega = \Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k}$ and expanding the cross product yields

$$\mathbf{M} = \left( (\dot{\mathbf{H}}_x)_{xyz} - \Omega_z \mathbf{H}_y + \Omega_y \mathbf{H}_z \right) \mathbf{i} + \left( (\dot{\mathbf{H}}_y)_{xyz} - \Omega_z \mathbf{H}_x + \Omega_x \mathbf{H}_z \right) \mathbf{j} + \left( (\dot{\mathbf{H}}_z)_{xyz} - \Omega_y \mathbf{H}_x + \Omega_x \mathbf{H}_y \right) \mathbf{k}$$

Substitute $\mathbf{H}_x, \mathbf{H}_y$, and $\mathbf{H}_z$ using Eq. 21–10. For the i component,

$$\Sigma M_i = \frac{d}{dt}(I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z) - \Omega_z (I_y \omega_x - I_{yz} \omega_z - I_{yx} \omega_y) + \Omega_y (I_z \omega_z - I_{zx} \omega_x - I_{zy} \omega_y) \quad \text{Ans.}$$

One can obtain y and z components in a similar manner.

**Ans:**

$$\Sigma M_i = \frac{d}{dt}(I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z) - \Omega_z (I_y \omega_x - I_{yz} \omega_z - I_{yx} \omega_y) + \Omega_y (I_z \omega_z - I_{zx} \omega_x - I_{zy} \omega_y)$$
21–41.

Derive the scalar form of the rotational equation of motion about the x axis if \( \Omega \neq \omega \) and the moments and products of inertia of the body are constant with respect to time.

**SOLUTION**

In general

\[
M = \frac{d}{dt} (H_x i + H_y j + H_z k)
\]

\[
= \left( \dot{H}_x i + \dot{H}_y j + \dot{H}_z k \right)_{xyz} + \Omega \times (H_x i + H_y j + H_z k)
\]

Substitute \( \Omega = \Omega_x i + \Omega_y j + \Omega_z k \) and expanding the cross product yields

\[
M = \left( (\dot{H}_x)_{xyz} - \Omega_z H_y + \Omega_y H_z \right) i + \left( (\dot{H}_y)_{xyz} - \Omega_x H_z + \Omega_z H_x \right) j + \left( (\dot{H}_z)_{xyz} - \Omega_y H_x + \Omega_x H_y \right) k
\]

Substitute \( H_x, H_y \) and \( H_z \) using Eq. 21–10. For the i component

\[
\sum M_x = \frac{d}{dt} (I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z) - \Omega_z (I_y \omega_y - I_{yz} \omega_z - I_{yx} \omega_x)
\]

\[
+ \Omega_y (I_z \omega_z - I_{zx} \omega_x - I_{zy} \omega_y)
\]

For constant inertia, expanding the time derivative of the above equation yields

\[
\sum M_x = (I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z) - \Omega_z (I_y \omega_y - I_{yz} \omega_z - I_{yx} \omega_x)
\]

\[
+ \Omega_y (I_z \omega_z - I_{zx} \omega_x - I_{zy} \omega_y)
\]

Ans.

One can obtain y and z components in a similar manner.
SOLUTION

In general

\[
\mathbf{M} = \frac{d}{dt} (H_x \mathbf{i} + H_y \mathbf{j} + H_z \mathbf{k})
\]

\[
= (\dot{H}_x \mathbf{i} + \dot{H}_y \mathbf{j} + \dot{H}_z \mathbf{k})_{xyz} + \mathbf{\Omega} \times (H_x \mathbf{i} + H_y \mathbf{j} + H_z \mathbf{k})
\]

Substitute \( \mathbf{\Omega} = \Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k} \) and expanding the cross product yields

\[
\mathbf{M} = \left( (\dot{H}_x)_{xyz} - \Omega_z H_y + \Omega_y H_z \right) \mathbf{i} + \left( (\dot{H}_y)_{xyz} - \Omega_z H_x + \Omega_x H_z \right) \mathbf{j}
\]

\[
+ \left( (\dot{H}_z)_{xyz} - \Omega_y H_x + \Omega_x H_y \right) \mathbf{k}
\]

Substitute \( H_x, H_y \) and \( H_z \) using Eq. 21–10. For the \( i \) component

\[
\sum M_x = \frac{d}{dt} (I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z) - \Omega_z (I_y \omega_y - I_{yz} \omega_z - I_{zy} \omega_x)
\]

\[
+ \Omega_y (I_z \omega_z - I_{zx} \omega_x - I_{zy} \omega_y)
\]

Set \( I_{xy} = I_{yx} = I_{zx} = 0 \) and require \( I_x, I_y, I_z \) to be constant. This yields

\[
\sum M_x = I_x \dot{\omega}_x - I_y \Omega_z \omega_y + I_z \Omega_y \omega_z \quad \text{Ans.}
\]

One can obtain \( y \) and \( z \) components in a similar manner.
21–43.

The 4-lb bar rests along the smooth corners of an open box. At the instant shown, the box has a velocity \( \mathbf{v} = \{3j\} \text{ ft/s} \) and an acceleration \( \mathbf{a} = \{-6j\} \text{ ft/s}^2 \). Determine the \( x, y, z \) components of force which the corners exert on the bar.

**SOLUTION**

\[
\begin{align*}
\sum F_x &= m(a_G)_x; \quad A_x + B_x = 0 \\
\sum F_y &= m(a_G)_y; \quad A_y + B_y = (\frac{4}{32.2})(-6) \\
\sum F_z &= m(a_G)_z; \quad B_z - 4 = 0 \quad B_z = 4 \text{ lb}
\end{align*}
\]

Applying Eq. 21–25 with \( \omega_x = \omega_y = \omega_z = 0 \), \( \dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0 \),

\[
\sum (M_G)_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z; \quad B_x(1) - A_x(1) + 4(0.5) = 0 \\
\sum (M_G)_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x; \quad A_y(1) - B_y(1) + 4(1) = 0
\]


\[
\begin{align*}
A_x &= -2.00 \text{ lb} \quad A_y = 0.627 \text{ lb} \quad B_x = 2.00 \text{ lb} \quad B_y = -1.37 \text{ lb} \\
\sum (M_G)_z &= I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y; \\
(-2.00)(0.5) - (2.00)(0.5) - (-1.37)(1) + (0.627)(1) &= 0 \quad \text{(O.K!)}
\end{align*}
\]

Ans:

\[
\begin{align*}
B_x &= 4 \text{ lb} \\
A_x &= -2.00 \text{ lb} \\
A_y &= 0.627 \text{ lb} \\
B_x &= 2.00 \text{ lb} \\
B_y &= -1.37 \text{ lb}
\end{align*}
\]
The uniform rectangular plate has a mass of \( m = 2 \text{ kg} \) and is given a rotation of \( \omega = 4 \text{ rad/s} \) about its bearings at \( A \) and \( B \). If \( a = 0.2 \text{ m} \) and \( c = 0.3 \text{ m} \), determine the vertical reactions at the instant shown. Use the \( x, y, z \) axes shown and note that \( I_{zx} = -\frac{mac_12}{c^2 + a^2} \).

**SOLUTION**

\[ \begin{align*}
\omega_x &= 0, \quad \omega_y = 0, \quad \omega_z = -4 \\
\dot{\omega}_x &= 0, \quad \dot{\omega}_y = 0, \quad \dot{\omega}_z = 0 \\
\sum M_y &= I_{yy} \omega_y - (I_{zz} - I_{xx}) \omega_z \omega_x - I_{yz} (\dot{\omega}_z - \omega_x \omega_y) \\
&= -I_{xx} (\omega_z^2 - \omega_x^2) - I_{yz} (\dot{\omega}_z + \omega_x \omega_y) \\
B_x - A_x &= \left( \frac{mac_1}{6} \right) \frac{c^2 - a^2}{a^2 + c^2} \omega^2 \\
\sum F_x &= m(a_c)_x; \quad A_x + B_x - mg = 0
\end{align*} \]

Substitute the data,

\[ B_x - A_x = \frac{2(0.2)(0.3)}{6} \left[ \frac{(0.3)^2 - (0.2)^2}{(0.3)^2 + (0.2)^2} \right] (-4)^2 = 0.34135 \]

\[ A_x + B_x = 2(9.81) \]

Solving:

\[ A_x = 9.64 \text{ N} \quad \text{Ans.} \]
\[ B_x = 9.98 \text{ N} \quad \text{Ans.} \]
21–45.

If the shaft $AB$ is rotating with a constant angular velocity of $\omega = 30 \text{ rad/s}$, determine the $X$, $Y$, $Z$ components of reaction at the thrust bearing $A$ and journal bearing $B$ at the instant shown. The disk has a weight of 15 lb. Neglect the weight of the shaft $AB$.

**SOLUTION**

The rotating $xyz$ frame is set with its origin at the plate's mass center, Fig. a. This frame will be fixed to the disk so that its angular velocity is $\Omega = \omega$ and the $x$, $y$, and $z$ axes will always be the principle axes of inertia of the disk. Referring to Fig. b,

$$\omega = [30 \cos 30^\circ - 30 \sin 30^\circ \mathbf{k}] \text{ rad/s} = [25.98 \mathbf{j} - 15 \mathbf{k}] \text{ rad/s}$$

Thus,

$$\omega_x = 0 \quad \omega_y = 25.98 \text{ rad/s} \quad \omega_z = -15 \text{ rad/s}$$

Since $\omega$ is always directed towards the $+Y$ axis and has a constant magnitude, $\dot{\omega} = 0$. Also, since $\dot{\Omega} = \omega$, $(\dot{\omega}_{xyz}) = \dot{\omega} = 0$. Thus,

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

The mass moments of inertia of the disk about the $x$, $y$, $z$ axes are

$$I_x = I_z = \frac{1}{4} \left( \frac{15}{32.2} \right) (0.5^2) = 0.02911 \text{ slug \cdot ft}^2$$

$$I_y = \frac{1}{2} \left( \frac{15}{32.2} \right) (0.5^2) = 0.05823 \text{ slug \cdot ft}^2$$

Applying the equations of motion,

$$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z) \omega_y \omega_z; \quad B_Z(1) - A_Z(1.5) = 0 - (0.05823 - 0.02911)(25.98)(-15)$$

$$B_Z - 1.5A_Z = 11.35 \quad (1)$$

$$\Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x) \omega_x \omega_z; \quad B_X(1 \sin 30^\circ) - A_X(1.5 \sin 30^\circ) = 0 - 0$$

$$B_X - 1.5A_X = 0 \quad (2)$$

$$\Sigma M_z = I_z \dot{\omega}_z - (I_y - I_x) \omega_x \omega_y; \quad B_X(1 \cos 30^\circ) - A_X(1.5 \cos 30^\circ) = 0 - 0$$

$$B_X - 1.5A_X = 0 \quad (3)$$

$$\Sigma F_x = m(a_G)X; \quad A_X + B_X = 0$$

$$A_X = B_X = 0 \quad (4)$$

$$\Sigma F_y = m(a_G)Y; \quad A_Y = 0$$

$$A_Y = 0$$

$$\Sigma F_z = m(a_G)Z; \quad A_Z + B_Z - 15 = 0$$

$$A_Z = 1.461 \text{ lb} \quad B_Z = 13.54 \text{ lb} = 13.5 \text{ lb} \quad \text{Ans.}$$

$$A_X = B_X = 0 \quad \text{Ans.}$$

Ans:

$$A_Z = 1.46 \text{ lb}$$

$$B_Z = 13.5 \text{ lb}$$

$$A_X = A_Y = B_X = 0$$

1154
21–46.

The assembly is supported by journal bearings at $A$ and $B$, which develop only $y$ and $z$ force reactions on the shaft. If the shaft is rotating in the direction shown at $\omega = 2\text{ }\text{rad/s}$, determine the reactions at the bearings when the assembly is in the position shown. Also, what is the shaft’s angular acceleration? The mass per unit length of each rod is $5\text{ kg/m}$.

**SOLUTION**

*Equations of Motion.* The inertia properties of the assembly are

$$I_x = \frac{1}{3} [5(1)](1^2) = 1.6667\text{ kg} \cdot \text{m}^2$$

$$I_y = \frac{1}{3} [5(3)](3^2) + [0 + 5(1)(1^2)] = 50.0\text{ kg} \cdot \text{m}^2$$

$$I_z = \frac{1}{3} [5(3)](3^2) + \left[ \frac{1}{12} [5(1)](1^2) + 5(1)(1^2 + 0.5^2) \right] = 51.67\text{ kg} \cdot \text{m}^2$$

$$I_{xy} = 0 + [5(1)](1)(0.5) = 2.50\text{ kg} \cdot \text{m}^2 \quad I_{yz} = I_{zx} = 0$$

with $\dot{\omega}_x = 2\text{ rad/s}$, $\omega_y = \omega_z = 0$ and $\ddot{\omega}_y = \ddot{\omega}_z = 0$ by referring to the FBD of the assembly, Fig. a,

$$\Sigma M_x = I_x \ddot{\omega}_x; \quad -[5(1)](9.81)(0.5) = 1.6667 \ddot{\omega}_x$$

$$\dot{\omega}_x = -14.715 \text{ rad/s}^2 = -14.7 \text{ rad/s}^2 \quad \text{Ans.}$$

$$\Sigma M_y = -I_{xy} \dot{\omega}_x; \quad [5(1)](9.81)(1) + [5(2)](9.81)(1.5) - B_z(3) = -2.50(-14.715)$$

$$B_z = 77.6625\text{ N} = 77.7\text{ N} \quad \text{Ans.}$$

$$\Sigma M_z = -I_{yz} \dot{\omega}_z; \quad -B_y(3) = -2.50(2^3) \quad B_y = 3.3333\text{ N} = 3.33\text{ N} \quad \text{Ans.}$$

$$\Sigma F_x = M(a_G)_x; \quad A_x = 0 \quad \text{Ans.}$$

$$\Sigma F_y = M(a_G)_y; \quad -A_y - 3.333 = [5(1)][-2^2(0.5)] \quad A_y = 6.667\text{ N} = 6.67\text{ N} \quad \text{Ans.}$$

$$\Sigma F_z = M(a_G)_z; \quad A_z + 77.6625 - [5(3)](9.81) - [5(1)](9.81) = [5(1)](-14.715)(0.5)$$

$$A_z = 81.75\text{ N} \quad \text{Ans.}$$

Ans:

$$\dot{\omega}_x = -14.7 \text{ rad/s}^2$$

$$B_z = 77.7\text{ N}$$

$$B_y = 3.33\text{ N}$$

$$A_y = 6.67\text{ N}$$

$$A_z = 81.75\text{ N}$$
21–47.

The assembly is supported by journal bearings at \( A \) and \( B \), which develop only \( y \) and \( z \) force reactions on the shaft. If the shaft \( A \) is subjected to a couple moment \( \mathbf{M} = 40 \mathbf{i} \) N⋅m, and at the instant shown the shaft has an angular velocity of \( \omega = 2 \) rad/s, determine the reactions at the bearings of the assembly at this instant. Also, what is the shaft’s angular acceleration? The mass per unit length of each rod is 5 kg/m.

**SOLUTION**

*Equations of Motions.* The inertia properties of the assembly are

\[
I_x = \frac{1}{3}[5(1)](1^2) = 1.6667 \text{ kg} \cdot \text{m}^2
\]

\[
I_y = \frac{1}{3}[5(3)](3^2) + \left[ 0 + [5(1)](1^2) \right] = 50.0 \text{ kg} \cdot \text{m}^2
\]

\[
I_z = \frac{1}{3}[5(3)](3^2) + \left\{ \frac{1}{12} [5(1)](1^2) + [5(1)](1^2 + 0.5^2) \right\} = 51.67 \text{ kg} \cdot \text{m}^2
\]

\[I_{xy} = 0 + [5(1)](1)(0.5) = 2.50 \text{ kg} \cdot \text{m}^2\]

\[I_{yz} = I_{zx} = 0\]

with \( \omega_x = 2 \) rad/s, \( \omega_y = \omega_z = 0 \) and \( \dot{\omega}_y = \dot{\omega}_z = 0 \) by referring to the FBD of the assembly, Fig. a,

\[
\Sigma M_x = I_x \dot{\omega}_x; \quad 40 - [5(1)](9.81)(0.5) = 1.6667 \dot{\omega}_x
\]

\[\dot{\omega}_x = 9.285 \text{ rad/s}^2\quad \text{Ans.}\]

\[
\Sigma M_y = -I_{xy} \dot{\omega}_x; \quad [5(1)](9.81)(1) + [5(3)](9.81)(1.5) - B_z(3) = -2.50(9.285)
\]

\[B_z = 97.6625 \text{ N} = 97.7 \text{ N}\quad \text{Ans.}\]

\[
\Sigma M_z = -I_{xz} \dot{\omega}_x; \quad -B_y(3) = -2.50(2^2) \quad B_y = 3.3333 \text{ N} = 3.33 \text{ N}\quad \text{Ans.}\]

\[
\Sigma F_x = M(a_G)\dot{\omega}_x; \quad A_x = 0
\]

\[
\Sigma F_y = M(a_G)\dot{\omega}_x; \quad -A_y - 3.3333 = [5(1)]\left[ -2^2(0.5) \right] \quad A_y = 6.6667 \text{ N} = 6.67 \text{ N}\quad \text{Ans.}\]

\[
\Sigma F_z = M(a_G)\dot{\omega}_x; \quad A_z + 97.6625 - [5(3)](9.81) - [5(1)](9.81) = [5(1)](9.285)(0.5)
\]

\[A_z = 121.75 \text{ N} = 122 \text{ N}\quad \text{Ans.}\]

\[
\dot{\omega}_x = 9.285 \text{ rad/s}^2\quad \text{Ans.}\]

\[B_z = 97.7 \text{ N}\]

\[B_y = 3.33 \text{ N}\]

\[A_y = 6.67 \text{ N}\]

\[A_z = 122 \text{ N}\]

The man sits on a swivel chair which is rotating with a constant angular velocity of \( 3 \text{ rad/s} \). He holds the uniform 5-lb rod \( AB \) horizontal. He suddenly gives it an angular acceleration of \( 2 \text{ rad/s}^2 \), measured relative to him, as shown. Determine the required force and moment components at the grip, \( A \), necessary to do this. Establish axes at the rod’s center of mass \( G \), with \(+z\) upward, and \(+y\) directed along the axis of the rod towards \( A \).

**SOLUTION**

\[
I_x = I_z = \frac{1}{12}(\frac{5}{32.2})(3)^2 = 0.1165 \text{ ft}^4 \\
I_y = 0 \\
\Omega = \omega = 3k \\
\omega_x = \omega_y = 0 \\
\omega_z = 3 \text{ rad/s} \\
\dot{\Omega} = (\dot{\omega}_{xyz}) + \Omega \times \omega = -2i + 0 \\
\dot{\omega}_x = -2 \text{ rad/s}^2 \\
\dot{\omega}_y = \dot{\omega}_z = 0 \\
(a_G)_x = 0 \\
(a_G)_y = (3.5)(3)^2 = 31.5 \text{ ft/s}^2 \\
(a_G)_z = 2(1.5) = 3 \text{ ft/s}^2 \\
\Sigma F_x = m(a_G)_x; \quad A_x = 0 \quad \text{Ans.} \\
\Sigma F_y = m(a_G)_y; \quad A_y = \frac{5}{32.2}(3.5) = 4.89 \text{ lb} \quad \text{Ans.} \\
\Sigma F_z = m(a_G)_z; \quad -5 + A_z = \frac{5}{32.2}(3) \\
A_z = 5.47 \text{ lb} \quad \text{Ans.} \\
\Sigma M_x = I_y(\dot{\omega}_x) - (I_y - I_z)\omega_y \omega_z; \quad \\
M_x = 5.47(1.5) = 0.1165(-2) = 0 \quad \text{Ans.} \\
\Sigma M_y = I_x(\dot{\omega}_y) - (I_z - I_x)\omega_x \omega_z; \quad \\
0 + M_y = 0 - 0 \quad \text{Ans.} \\
M_y = 0 \\
\Sigma M_z = I_z(\dot{\omega}_z) - (I_x - I_y)\omega_x \omega_y; \\
M_z = 0 - 0 \quad \text{Ans.} \\
M_z = 0 \\
M_x = 8.43 \text{ lb} \cdot \text{ft} \\
M_y = -8.43 \text{ lb} \cdot \text{ft} \\
M_z = 0 \\
M_x = 0 \\
M_y = 0 \\
M_z = 0 \\
\text{Ans:} \\
A_x = 0 \\
A_y = 4.89 \text{ lb} \\
A_z = 5.47 \text{ lb} \\
M_x = -8.43 \text{ lb} \cdot \text{ft} \\
M_y = 0 \\
M_z = 0
21–49.

The rod assembly is supported by a ball-and-socket joint at C and a journal bearing at D, which develops only x and y force reactions. The rods have a mass of 0.75 kg/m. Determine the angular acceleration of the rods and the components of reaction at the supports at the instant \( \omega = 8 \text{ rad/s} \) as shown.

**SOLUTION**

\[ \Omega = \omega = 8 \text{k} \]

\[ \omega_x = \omega_y = 0, \quad \omega_z = 8 \text{ rad/s} \]

\[ \dot{\omega}_x = \dot{\omega}_y = 0, \quad \dot{\omega}_z = \dot{\omega}_z \]

\[ I_{xz} = I_{xy} = 0 \]

\[ I_{yz} = 0.75(1)(2)(0.5) = 0.75 \text{ kg} \cdot \text{m}^2 \]

\[ I_{zz} = \frac{1}{3}(0.75)(1)(1)^2 = 0.25 \text{ kg} \cdot \text{m}^2 \]

Eqs. 21–24 become

\[ \Sigma M_x = I_{yz} \dot{\omega}_z^2 \]

\[ \Sigma M_y = -I_{yz} \dot{\omega}_z \]

\[ \Sigma M_z = I_{zz} \dot{\omega}_z \]

Thus,

\[ -D_z(4) - 7.3575(0.5) = 0.75(8)^2 \]

\[ D_y = -12.9 \text{ N} \quad \text{Ans.} \]

\[ D_x(4) = -0.75 \dot{\omega}_z \]

\[ 50 = 0.25 \dot{\omega}_z \]

\[ \dot{\omega}_z = 200 \text{ rad/s}^2 \quad \text{Ans.} \]

\[ D_x = -37.5 \text{ N} \quad \text{Ans.} \]

\[ \Sigma F_x = m(a_G)_x; \quad C_x - 37.5 = -1(0.75)(200)(0.5) \]

\[ C_x = -37.5 \text{ N} \quad \text{Ans.} \]

\[ \Sigma F_y = m(a_G)_y; \quad C_y - 12.9 = -1(0.75)(8)^2(0.5) \]

\[ C_y = -11.1 \text{ N} \quad \text{Ans.} \]

\[ \Sigma F_z = m(a_G)_z; \quad C_z - 7.3575 - 29.43 = 0 \]

\[ C_z = 36.8 \text{ N} \quad \text{Ans.} \]

\[ \dot{\omega}_z = 200 \text{ rad/s}^2 \]

\[ D_y = -12.9 \text{ N} \]

\[ D_x = -37.5 \text{ N} \]

\[ C_x = -37.5 \text{ N} \]

\[ C_y = -11.1 \text{ N} \]

\[ C_z = 36.8 \text{ N} \]
21–50.

The bent uniform rod $ACD$ has a weight of 5 lb/ft and is supported at $A$ by a pin and at $B$ by a cord. If the vertical shaft rotates with a constant angular velocity $\omega = 20$ rad/s, determine the $x$, $y$, $z$ components of force and moment developed at $A$ and the tension in the cord.

**SOLUTION**

$w_x = w_y = 0$

$w_z = 20$ rad/s

$\dot{w}_x = \dot{w}_y = \dot{w}_z = 0$

The center of mass is located at

$$\bar{z} = \frac{5(1)(\frac{1}{2})}{5(2)} = 0.25 \text{ ft}$$

$$\bar{y} = 0.25 \text{ ft (symmetry)}$$

$$I_{yc} = \frac{5}{32.2}(-0.5)(1) = -0.0776 \text{ slug} \cdot \text{ft}^2$$

$$I_{cz} = 0$$

Eqs. 21–24 reduce to

$$\Sigma M_x = I_{cz}(w_z)^2;$$

$$-T_B(0.5) - 10(0.75) = -0.0776(20)^2$$

$T_B = 47.1 \text{ lb}$

$$\Sigma M_y = 0; \quad M_y = 0$$

$$\Sigma M_z = 0; \quad M_z = 0$$

$$\Sigma F_x = ma_x; \quad A_x = 0$$

$$\Sigma F_y = ma_y; \quad A_y = -\left(\frac{10}{32.2}\right)(20)^2(1 - 0.25)$$

$A_y = -93.2 \text{ lb}$

$$\Sigma F_z = ma_z; \quad A_z - 47.1 - 10 = 0$$

$A_z = 57.1 \text{ lb}$

Ans: $T_B = 47.1 \text{ lb}$

Ans: $M_y = 0$

Ans: $M_z = 0$

Ans: $A_x = 0$

Ans: $A_y = -93.2 \text{ lb}$

Ans: $A_z = 57.1 \text{ lb}$
21–51.

The uniform hatch door, having a mass of 15 kg and a mass center at \( G \), is supported in the horizontal plane by bearings at \( A \) and \( B \). If a vertical force \( F = 300 \) N is applied to the door as shown, determine the components of reaction at the bearings and the angular acceleration of the door. The bearing at \( A \) will resist a component of force in the \( y \) direction, whereas the bearing at \( B \) will not. For the calculation, assume the door to be a thin plate and neglect the size of each bearing. The door is originally at rest.

**SOLUTION**

\[
\omega_x = \omega_y = \omega_z = 0
\]
\[
\dot{\omega}_x = \dot{\omega}_z = 0
\]

Eqs. 21–25 reduce to

\[
\Sigma M_x = 0; \quad 300(0.25 - 0.03) + B_z(0.15) - A_z(0.15) = 0
\]
\[
B_z - A_z = -440
\]  
(1)

\[
\Sigma M_y = I_z \dot{\omega}_z; \quad 15(9.81)(0.2) - (300)(0.4 - 0.03) = \frac{1}{12}(15)(0.4)^2 + 15(0.2)^2 \dot{\omega}_y
\]
\[
\dot{\omega}_y = -102 \text{ rad/s}^2
\]  
Ans.

\[
\Sigma M_z = 0; \quad -B_x(0.15) + A_x(0.15) = 0
\]
\[
\Sigma F_x = m(a_G)_x; \quad -A_x + B_x = 0
\]
\[
A_x = B_x = 0
\]  
Ans.

\[
\Sigma F_y = m(a_G)_y; \quad A_y = 0
\]  
Ans.

\[
\Sigma F_z = m(a_G)_z; \quad 300 - 15(9.81) + B_z + A_z = 15(101.96)(0.2)
\]
\[
B_z + A_z = 153.03
\]  
(2)

Solving Eqs. (1) and (2) yields

\[
A_z = 297 \text{ N} \quad \text{Ans.}
\]
\[
B_z = -143 \text{ N} \quad \text{Ans.}
\]

Ans:

\[
\omega_z = -102 \text{ rad/s}^2
\]
\[
A_x = B_x = 0
\]
\[
A_y = 0
\]
\[
A_z = 297 \text{ N}
\]
\[
B_z = -143 \text{ N}
\]
*21–52.

The 5-kg circular disk is mounted off center on a shaft which is supported by bearings at A and B. If the shaft is rotating at a constant rate of \( \omega = 10 \text{ rad/s} \), determine the vertical reactions at the bearings when the disk is in the position shown.

**SOLUTION**

\[
\begin{align*}
\omega_x &= 0, \quad \omega_y = -10 \text{ rad/s}, \quad \omega_z = 0 \\
\dot{\omega}_x &= 0, \quad \dot{\omega}_y = 0, \quad \dot{\omega}_z = 0 \\
\zeta + \Sigma M_x &= I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z \\
-0.2(F_A) + (0.2)(F_B) &= 0 \\
F_A &= F_B \\
+\Sigma F_z &= ma_z: \quad F_A + F_B - 5(9.81) = -5(10)^2 (0.02) \\
F_A &= F_B = 19.5 \text{ N}
\end{align*}
\]

*Ans:*

\( F_A = F_B = 19.5 \text{ N} \)
Two uniform rods, each having a weight of 10 lb, are pin connected to the edge of a rotating disk. If the disk has a constant angular velocity \( \omega_D = 4 \) rad/s, determine the angle \( \theta \) made by each rod during the motion, and the components of the force and moment developed at the pin \( A \). Suggestion: Use the \( x, y, z \) axes oriented as shown.

**SOLUTION**

\[
I_y = \frac{1}{12} \left( \frac{10}{32.2} \right)(4)^2 = 0.4141 \text{ slug} \cdot \text{ft}^2 \\
I_z = 0
\]

Applying Eq. 21–25 with

\[
\omega_y = 4 \sin \theta \\
\omega_z = 4 \cos \theta \\
\omega_x = 0
\]

\[
\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0
\]

\[
\sum M_x = I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z; \\
\sum M_y = I_y \dot{\omega}_y - (I_z - I_x)\omega_z \omega_x; \\
\sum M_z = I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y; \\
M_x + A_y(2) = 0 \quad \text{Ans.}
\]

Also,

\[
\sum F_x = m(a_G)_x; \\
A_x = 0 \quad \text{Ans.}
\]

From Eq. (2)

\[
M_y = 0 \quad \text{Ans.}
\]

\[
\sum F_y = m(a_G)_y; \\
A_y - 10 \sin \theta = -\left( \frac{10}{32.2} \right)(1.75 + 2 \sin \theta)(4)^2 \cos \theta \quad \text{(3)}
\]

\[
\sum F_z = m(a_G)_z; \\
A_z - 10 \cos \theta = \left( \frac{10}{32.2} \right)(1.75 + 2 \sin \theta)(4)^2 \sin \theta \quad \text{(4)}
\]

Solving Eqs. (1), (3) and (4) yields:

\[
\theta = 64.1^\circ \\
A_y = 1.30 \text{ lb} \\
A_z = 20.2 \text{ lb} \quad \text{Ans.}
\]

**Ans:**

\[
M_x = 0 \\
A_x = 0 \\
M_y = 0 \\
\theta = 64.1^\circ \\
A_y = 1.30 \text{ lb} \\
A_z = 20.2 \text{ lb}
\]
The 10-kg disk turns around the shaft $AB$, while the shaft rotates about $BC$ at a constant rate of $\omega_x = 5 \text{ rad/s}$. If the disk does not slip, determine the normal and frictional force it exerts on the ground. Neglect the mass of shaft $AB$.

**SOLUTION**

**Kinematics.** The instantaneous axis of zero velocity ($IA$) is indicated in Fig. a. Here the resultant angular velocity is always directed along $IA$. The fixed reference frame is set to coincide with the rotating $xyz$ frame using the similar triangle,

$$\frac{\omega_z}{2} = \frac{\omega_x}{0.4}; \quad \omega_z = \frac{2}{0.4} (5) = 25.0 \text{ rad/s}$$

Thus,

$$\omega = \omega_x + \omega_z = \{-5i + 25.0k\} \text{ rad/s}$$

Here, $(\dot{\omega}_x)_{xyz} = (\dot{\omega}_z)_{xyz} = 0$ since $\omega_x$ is constant. The direction of $\omega_x$ will not change that always along $x$ axis when $\Omega = \omega_x$. Then

$$\omega_x = (\dot{\omega}_x)_{xyz} + \omega_x \times \omega_x = 0$$

The direction of $\omega_x$ does not change with reference to the $xyz$ rotating frame if this frame rotates with $\Omega = \omega_x = \{-5i\} \text{ rad/s}$. Then

$$\omega_z = (\dot{\omega}_z)_{xyz} + \omega_x \times \omega_z$$

$$= 0 + (-5i) \times (25.0k)$$

$$= \{125j\} \text{ rad/s}^2$$

Finally

$$\omega = \omega_x + \omega_z = 0 + 125j = \{125j\} \text{ rad/s}^2$$
Equations of Motion. The mass moments of inertia of the disk about the \( x \), \( y \) and \( z \) axes are

\[
I_x = I_y = \frac{1}{4}(10)(0.4^2) + 10(2^2) = 40.4 \text{ kg} \cdot \text{m}^2
\]

\[
I_z = \frac{1}{2}(10)(0.4^2) = 0.800 \text{ kg} \cdot \text{m}^2
\]

By referring to the FBD of the disk, Fig. b, \( \Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x)\omega_x \omega_y; \)

\[
N(2) - 10(9.81)(2) = 40.4(125) - (0.8 - 40.4)(25.0)(-5)
\]

\[
N = 148.1 \text{ N} = 148 \text{ N} \quad \text{Ans.}
\]

\[
\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y; \quad F_f(0.4) = 0 - 0
\]

\[
F_f = 0 \quad \text{Ans.}
\]

Ans:

\[
N = 148 \text{ N} \\
F_f = 0
\]
21–55.

The 20-kg disk is spinning on its axle at $\omega_y = 30 \text{ rad/s}$, while the forked rod is turning at $\omega_1 = 6 \text{ rad/s}$. Determine the $x$ and $z$ moment components the axle exerts on the disk during the motion.

**SOLUTION**

**Solution I**

**Kinematics.** The fixed reference $XYZ$ frame is set coincident with the rotating $xyz$ frame. Here, this rotating frame is set to rotate with $\Omega = \omega_p = \{-6\} \text{ rad/s}$. The angular velocity of the disk with respect to the $XYZ$ frame is

$$\omega = \omega_p + \omega_s = \{30\hat{j} - 6\hat{k}\} \text{ rad/s}$$

Then

$$\omega_x = 0 \quad \omega_y = 30 \text{ rad/s} \quad \omega_z = -6 \text{ rad/s}$$

Since $\omega_p$ and $\omega_s$ does not change with respect to $xyz$ frame, $\dot{\omega}$ with respect to this frame is $\dot{\omega} = 0$. Then

$$\dot{\omega}_x = 0 \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

**Equation of Motion.** Although the disk spins about the $y$ axis, but the mass moment of inertia of the disk remain constant with respect to the $xyz$ frame.

$$I_x = I_z = \frac{1}{4}(20)(0.2^2) = 0.2 \text{ kg} \cdot \text{m}^2$$

$$I_y = \frac{1}{2}(20)(0.2^2) = 0.4 \text{ kg} \cdot \text{m}^2$$
With $\Omega \neq \omega$ with $\Omega_x = 0$, $\Omega_y = 0$ and $\Omega_z = -6$ rad/s

$$\Sigma M_x = I_x \dot{\omega}_x = I_x \Omega_x \omega_y + I_x \Omega_y \omega_z;$$

$$(M_0)_x = 0 - 0.4(-6)(30) + 0 \quad (M_0)_x = 72.0 \text{ N} \cdot \text{m} \quad \text{Ans.}$$

$$\Sigma M_y = I_y \dot{\omega}_y = I_y \Omega_y \omega_z + I_y \Omega_z \omega_x;$$

$$0 = 0 \quad \text{(Satisfied!)}$$

$$\Sigma M_z = I_z \dot{\omega}_z = I_z \Omega_z \omega_x + I_z \Omega_x \omega_y;$$

$$(M_0)_z = 0 - 0 + 0 = 0 \quad \text{Ans.}$$

**Solution II**

Here, the $xyz$ frame is set to rotate with $\Omega = \omega = \{30\hat{j} - 6\hat{k}\}$ rad/s. Setting another $x'y'z'$ frame coincide with $xyz$ and $XYZ$ frame to have an angular velocity of $\Omega' = \omega_p = \{-6\hat{k}\}$ rad/s,

$$\dot{\omega} = (\dot{\omega})_{xyz} = (\dot{\omega})_{x'y'z'} + \Omega' \times \omega$$

$$= 0 + (-6\hat{k}) \times (30\hat{j} - 6\hat{k}) = \{180\hat{i}\} \text{ rad/s}^2$$

Thus

$$\dot{\omega}_x = 180 \text{ rad/s} \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

with $\Omega = \omega$,

$$\Sigma M_x = I_x \dot{\omega}_x = (I_y - I_z)\omega_y \omega_z; \quad (M_0)_x = 0.2(180) - (0.4 - 0.2)(30)(-6)$$

$$= 72.0 \text{ N} \cdot \text{m}$$

$$\Sigma M_y = I_y \dot{\omega}_y = (I_z - I_x)\omega_z \omega_x; \quad 0 = 0 - 0 \quad \text{(Satisfied)}$$

$$\Sigma M_z = I_z \dot{\omega}_z = (I_x - I_y)\omega_x \omega_y \quad (M_0)_z = 0 - 0 = 0 \quad \text{Ans.}$$

**Ans:**

$$(M_0)_x = 72.0 \text{ N} \cdot \text{m}$$

$$(M_0)_z = 0$$
*21–56.

The 4-kg slender rod $AB$ is pinned at $A$ and held at $B$ by a cord. The axle $CD$ is supported at its ends by ball-and-socket joints and is rotating with a constant angular velocity of 2 rad/s. Determine the tension developed in the cord and the magnitude of force developed at the pin $A$.

**SOLUTION**

$$ I_z = \frac{1}{3} (4)(2)^2 = 5.3333 \text{ kg} \cdot \text{m}^2 $$

Applying the third of Eq. 21–25 with

$$ \begin{align*}
\omega_x &= 2 \cos 40^\circ = 1.5321 \text{ rad/s} \\
\omega_z &= 2 \sin 40^\circ = 1.2856 \text{ rad/s} \\
\Sigma M_x &= I_y \omega_x - (I_y - I_z) \omega_x \omega_z; \\
T(2 \cos 40^\circ) - 4(9.81)(1 \sin 40^\circ) &= 0 - (0 - 5.3333)(1.5321)(1.2856) \\
T &= 23.3 \text{ N} \quad \text{Ans.}
\end{align*} $$

Also,

$$ \begin{align*}
\Sigma F_x &= m(a_G)_x; \quad A_x = 0 \\
\Sigma F_y &= m(a_G)_y; \quad A_y - 23.32 = -4(2)^2 (1 \sin 40^\circ) \quad A_y = 13.03 \text{ N} \\
\Sigma F_z &= m(a_G)_z; \quad A_z - 4(9.81) = 0 \quad A_z = 39.24 \text{ N} \\
F_A &= \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{0^2 + 13.03^2 + 39.24^2} = 41.3 \text{ N} \quad \text{Ans.}
\end{align*} $$
The blades of a wind turbine spin about the shaft $S$ with a constant angular speed of $\omega_p$, while the frame precesses about the vertical axis with a constant angular speed of $\omega_p$. Determine the $x$, $y$, and $z$ components of moment that the shaft exerts on the blades as a function of $\theta$. Consider each blade as a slender rod of mass $m$ and length $l$.

**SOLUTION**

The rotating $xyz$ frame shown in Fig. a will be attached to the blade so that it rotates with an angular velocity of $\Omega = \omega$, where $\omega = \omega_x^b + \omega_p$. Referring to Fig. b $\omega_p = \omega_p \sin \theta \mathbf{i} + \omega_p \cos \theta \mathbf{k}$. Thus, $\omega = \omega_p \sin \theta \mathbf{i} + \omega_s \mathbf{j} + \omega_p \cos \theta \mathbf{k}$. Then

$$\omega_x = \omega_p \sin \theta \quad \omega_y = \omega_s \quad \omega_z = \omega_p \cos \theta$$

The angular acceleration of the blade $\dot{\omega}$ with respect to the $XYZ$ frame can be obtained by setting another $x'y'z'$ frame having an angular velocity of $\Omega' = \omega_p = \omega_p \sin \theta \mathbf{i} + \omega_p \cos \theta \mathbf{k}$. Thus,

$$\dot{\omega} = (\dot{\omega})_{x'y'z'} + \Omega' \times \omega$$

$$= (\dot{\omega}_x)_{x'y'z'} + (\dot{\omega}_y)_{x'y'z'} + \Omega' \times \omega_x + \Omega' \times \omega_y + \Omega' \times \omega_z$$

$$= 0 + 0 + (\omega_p \sin \theta \mathbf{i} + \omega_p \cos \theta \mathbf{k}) \times (\omega_p \sin \theta \mathbf{i} + \omega_p \cos \theta \mathbf{k}) + 0$$

$$= -\omega_p \omega_p \cos \theta \mathbf{i} + \omega_p \omega_p \sin \theta \mathbf{k}$$

Since $\Omega = \omega, \dot{\omega}_{x'y'z'} = \dot{\omega}$. Thus,

$$\dot{\omega}_x = -\omega_p \omega_p \cos \theta \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = \omega_p \omega_p \sin \theta$$

Also, the $x$, $y$, and $z$ axes will remain as principle axes of inertia for the blade. Thus,

$$I_x = I_y = \frac{1}{12} (2m)(2l)^2 = \frac{2}{3} ml^2 \quad I_z = 0$$

Applying the moment equations of motion and referring to the free-body diagram shown in Fig. a,

$$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z) \omega_y \omega_z; \quad M_x = \frac{2}{3} ml^2 (-\omega_p \omega_p \cos \theta) - \left(\frac{2}{3} ml^2 - 0\right)(\omega_p \omega_p \cos \theta)$$

$$= -\frac{4}{3} ml^2 \omega_p \omega_p \cos \theta \quad \text{Ans.}$$

$$\Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x) \omega_x \omega_z; \quad M_y = 0 - \left(0 - \frac{4}{3} ml^2 \right) (\omega_p \omega_p \cos \theta) \omega_p \sin \theta$$

$$= \frac{1}{3} ml^2 \omega_p^2 \sin 2\theta \quad \text{Ans.}$$

$$\Sigma M_z = I_z \dot{\omega}_z - (I_y - I_x) \omega_x \omega_y; \quad M_z = 0 - 0 = 0 \quad \text{Ans.}$$

**Ans:**

$$M_x = -\frac{4}{3} ml^2 \omega_p \omega_p \cos \theta$$

$$M_y = \frac{1}{3} ml^2 \omega_p^2 \sin 2\theta$$

$$M_z = 0$$
21–58.

The 15-lb cylinder is rotating about shaft $AB$ with a constant angular speed $\omega = 4 \text{ rad/s}$. If the supporting shaft at $C$, initially at rest, is given an angular acceleration $a_C = 12 \text{ rad/s}^2$, determine the components of reaction at the bearings $A$ and $B$. The bearing at $A$ cannot support a force component along the $x$ axis, whereas the bearing at $B$ does.

**SOLUTION**

$$\omega = [-4i] \text{ rad/s}$$

$$\dot{\omega} = \dot{\omega}_{xyz} + \Omega \times \omega = 12k + 0 \times (-4i) = [12k] \text{ rad/s}^2$$

Hence

$$\omega_x = -4, \quad \omega_y = \omega_z = 0,$$

$$\dot{\omega}_x = \dot{\omega}_y = 0, \quad \dot{\omega}_z = 12 \text{ rad/s}^2$$

$$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z) \omega_y \omega_z, \quad 0 = 0$$

$$\Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x) \omega_z \omega_x, \quad B_z (1) - A_z (1) = 0$$

$$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y) \omega_x \omega_y, \quad A_y (1) - B_y (1) = \left[ 12 \left( \frac{15}{32.2} \right) \left( 3(0.5)^2 + (2)^2 \right) \right] (12) - 0$$

$$\Sigma F_x = m(a_C)_x; \quad B_x = 0$$

$$\Sigma F_y = m(a_C)_y; \quad A_y + B_y = -\left( \frac{15}{32.2} \right) (1)(12)$$

$$\Sigma F_z = m(a_C)_z; \quad A_z + B_z - 15 = 0$$

Solving,

$$A_y = -1.69 \text{ lb}$$

$$B_y = -3.90 \text{ lb}$$

$$A_z = B_z = 7.5 \text{ lb}$$

Ans: $B_x = 0$

Ans: $B_y = -3.90 \text{ lb}$

Ans: $A_y = -1.69 \text{ lb}$

Ans: $A_z = B_z = 7.5 \text{ lb}$
21–59.

The thin rod has a mass of 0.8 kg and a total length of 150 mm. It is rotating about its midpoint at a constant rate \( \dot{\theta} = 6 \) rad/s, while the table to which its axle \( A \) is fastened is rotating at \( 2 \) rad/s. Determine the \( x, y, z \) moment components which the axle exerts on the rod when the rod is in any position \( \theta \).

**SOLUTION**

The \( x,y,z \) axes are fixed as shown.

\[
\begin{align*}
\omega_x &= 2 \sin \theta \\
\omega_y &= 2 \cos \theta \\
\omega_z &= \dot{\theta} = 6 \\
\dot{\omega}_x &= 2\dot{\theta} \cos \theta = 12 \cos \theta \\
\dot{\omega}_y &= -2\dot{\theta} \sin \theta = -12 \sin \theta \\
\dot{\omega}_z &= 0 \\
I_x &= 0 \\
I_y &= I_z = \frac{1}{12}(0.8)(0.15)^2 = 1.5(10^{-3})
\end{align*}
\]

Using Eqs. 21–25:

\[
\begin{align*}
\Sigma M_x &= 0 - 0 = 0 \quad \text{Ans.} \\
\Sigma M_y &= 1.5(10^{-3})(-12 \sin \theta) - [1.5(10^{-3})-0](6)(2 \sin \theta) \\
\Sigma M_y &= (-0.036 \sin \theta) \text{ N} \cdot \text{m} \quad \text{Ans.} \\
\Sigma M_z &= 0 - [0 - 1.5(10^{-3})](2 \sin \theta)(2 \cos \theta) \\
\Sigma M_z &= 0.006 \sin \theta \cos \theta = (0.003 \sin 2\theta) \text{ N} \cdot \text{m} \quad \text{Ans.}
\end{align*}
\]

**Ans:**
\[
\begin{align*}
\Sigma M_x &= 0 \\
\Sigma M_y &= (-0.036 \sin \theta) \text{ N} \cdot \text{m} \\
\Sigma M_z &= (0.003 \sin 2\theta) \text{ N} \cdot \text{m}
\end{align*}
\]
*21–60.

Show that the angular velocity of a body, in terms of Euler angles $\phi$, $\theta$, and $\psi$, can be expressed as

$$
\omega = (\dot\phi \sin \theta \sin \psi + \dot\psi \cos \psi)\mathbf{i} + (\dot\phi \sin \theta \cos \psi - \dot\theta \sin \psi)\mathbf{j} + (\dot\phi \cos \theta + \dot\psi)\mathbf{k},
$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are directed along the $x$, $y$, $z$ axes as shown in Fig. 21–15d.

**SOLUTION**

From Fig. 21–15b, due to rotation $\phi$, the $x$, $y$, $z$ components of $\dot\phi$ are simply $\dot\phi$ along $z$ axis.

From Fig 21–15c, due to rotation $\theta$, the $x$, $y$, $z$ components of $\dot\phi$ and $\dot\theta$ are $\dot\phi \sin \theta$ in the $y$ direction, $\dot\phi \cos \theta$ in the $z$ direction, and $\dot\theta$ in the $x$ direction.

Lastly, rotation $\psi$. Fig. 21–15d, produces the final components which yields

$$
\omega = (\dot\phi \sin \theta \sin \psi + \dot\psi \cos \psi)\mathbf{i} + (\dot\phi \sin \theta \cos \psi - \dot\theta \sin \psi)\mathbf{j} + (\dot\phi \cos \theta + \dot\psi)\mathbf{k} \quad \text{Q.E.D.}
$$

Ans:

$$
\omega = (\dot\phi \sin \theta \sin \psi + \dot\psi \cos \psi)\mathbf{i} + (\dot\phi \sin \theta \cos \psi - \dot\theta \sin \psi)\mathbf{j} + (\dot\phi \cos \theta + \dot\psi)\mathbf{k}
$$
21–61.

A thin rod is initially coincident with the $Z$ axis when it is given three rotations defined by the Euler angles $\phi = 30^\circ$, $\theta = 45^\circ$, and $\psi = 60^\circ$. If these rotations are given in the order stated, determine the coordinate direction angles $\alpha$, $\beta$, $\gamma$ of the axis of the rod with respect to the $X$, $Y$, and $Z$ axes. Are these directions the same for any order of the rotations? Why?

**SOLUTION**

$$u = (\sin 45^\circ) \sin 30^\circ \mathbf{i} - (\sin 45^\circ) \cos 30^\circ \mathbf{j} + \cos 45^\circ \mathbf{k}$$

$$u = 0.3536 \mathbf{i} - 0.6124 \mathbf{j} + 0.5771 \mathbf{k}$$

$$\alpha = \cos^{-1} 0.3536 = 69.3^\circ$$  \hspace{1cm} \text{Ans.}

$$\beta = \cos^{-1}(-0.6124) = 128^\circ$$  \hspace{1cm} \text{Ans.}

$$\gamma = \cos^{-1}(0.7071) = 45^\circ$$  \hspace{1cm} \text{Ans.}

No, the orientation of the rod will not be the same for any order of rotation, because finite rotations are not vectors.
The gyroscope consists of a uniform 450-g disk $D$ which is attached to the axle $AB$ of negligible mass. The supporting frame has a mass of 180 g and a center of mass at $G$. If the disk is rotating about the axle at $\omega_D = 90 \text{ rad/s}$, determine the constant angular velocity $\omega_p$ at which the frame precesses about the pivot point $O$. The frame moves in the horizontal plane.

**SOLUTION**

\[ \Sigma M_x = I_z \Omega_y \omega_z \]

\[ (0.450)(9.81)(0.125) + (0.180)(9.81)(0.080) = \frac{1}{2} (0.450)(0.035)^2 \omega_p (90) \]

$\omega_p = 27.9 \text{ rad/s}$  

**Ans:**

\[ \omega_p = 27.9 \text{ rad/s} \]
21–63.

The toy gyroscope consists of a rotor \( R \) which is attached to the frame of negligible mass. If it is observed that the frame is precessing about the pivot point \( O \) at \( \omega_p = 2 \text{ rad/s} \), determine the angular velocity \( \omega_R \) of the rotor. The stem \( OA \) moves in the horizontal plane. The rotor has a mass of 200 g and a radius of gyration \( k_{OA} = 20 \text{ mm} \) about \( OA \).

**SOLUTION**

\[
\sum M_x = I_z \omega_y w_z
\]

\[
(0.2)(9.81)(0.03) = \left[ 0.2(0.02)^2 \right](2)(\omega_R)
\]

\[
\omega_R = 368 \text{ rad/s}
\]

**Ans.:** \( \omega_R = 368 \text{ rad/s} \)
The top consists of a thin disk that has a weight of 8 lb and a radius of 0.3 ft. The rod has a negligible mass and a length of 0.5 ft. If the top is spinning with an angular velocity \( \omega_s = 300 \text{ rad/s} \), determine the steady-state precessional angular velocity \( \omega_p \) of the rod when \( \theta = 40^\circ \).

**SOLUTION**

\[
\Sigma M_x = -I \dot{\phi}^2 \sin \theta \cos \theta + I_x \dot{\phi} \sin \theta (\dot{\phi} \cos \theta + \dot{\psi})
\]

\[
8(0.5 \sin 40^\circ) = -\left[ \frac{1}{4} \left( \frac{8}{32.2} \right)^2 + \left( \frac{8}{32.2} \right)(0.5)^2 \right] \omega_p^2 \sin 40^\circ \cos 40^\circ
\]

\[+ \left[ \frac{1}{2} \left( \frac{8}{32.2} \right)^2 \right] \omega_p \sin 40^\circ (\omega_p \cos 40^\circ + 300)
\]

\[0.02783 \omega_p^2 - 2.1559 \omega_p + 2.571 = 0
\]

\[\omega_p = 1.21 \text{ rad/s} \hspace{1cm} \text{Ans. (Low precession)}
\]

\[\omega_p = 76.3 \text{ rad/s} \hspace{1cm} \text{Ans. (High precession)}
\]
21–65.
Solve Prob. 21–64 when $\theta = 90^\circ$.  

**SOLUTION**  

\[ \Sigma M_x = I_z \Omega_y w_z \]

\[ 8(0.5) = \left[ \frac{1}{2} \cdot \frac{8}{32.2} \right] (0.3)^2 \cdot w_p(300) \]

\[ \omega_p = 1.19 \text{ rad/s} \quad \text{Ans.} \]
The propeller on a single-engine airplane has a mass of 15 kg and a centroidal radius of gyration of 0.3 m computed about the axis of spin. When viewed from the front of the airplane, the propeller is turning clockwise at 350 rad/s about the spin axis. If the airplane enters a vertical curve having a radius of 80 m and is traveling at 200 km/h, determine the gyroscopic bending moment which the propeller exerts on the bearings of the engine when the airplane is in its lowest position.

**SOLUTION**

\[ \omega_z = 350 \text{ rad/s} = \omega_z \]

\[ v = 200 \text{ km/h} = \frac{200(10^3)}{3600} = 55.56 \text{ m/s} \]

\[ \Omega_y = \frac{55.56}{80} = 0.694 \text{ rad/s} \]

\[ \sum M_x = I_x \Omega_y \omega_z \]

\[ M_x = [15(0.3)^2](0.694)(350) \]

\[ M_x = 328 \text{ N} \cdot \text{m} \]

**Ans:**

\[ M_x = 328 \text{ N} \cdot \text{m} \]
21–67.
A wheel of mass \( m \) and radius \( r \) rolls with constant spin \( \omega \) about a circular path having a radius \( a \). If the angle of inclination is \( \theta \), determine the rate of precession. Treat the wheel as a thin ring. No slipping occurs.

**SOLUTION**

Since no slipping occurs,

\[
(r) \dot{\psi} = (a + r \cos \theta) \dot{\phi}
\]

or

\[
\dot{\psi} = \left( \frac{a + r \cos \theta}{r} \right) \dot{\phi}
\]  \hspace{1cm} (1)

Also,

\[
\omega = \dot{\phi} + \dot{\psi}
\]

\[
\Sigma F_y = m(a\dot{\psi}); \quad F = m(\dot{a} \dot{\phi})
\]  \hspace{1cm} (2)

\[
\Sigma F_z = m(a\dot{\psi}); \quad N - mg = 0
\]  \hspace{1cm} (3)

\[
I_x = I_y = \frac{mr^2}{2}, \quad I_z = mr^2
\]

Thus,

\[
\omega_x = 0, \quad \omega_y = \dot{\phi} \sin \theta, \quad \omega_z = -\dot{\psi} + \dot{\phi} \cos \theta
\]

\[
\dot{\omega}_x = \dot{\phi} \psi \sin \theta, \quad \dot{\omega}_y = \dot{\phi} \cos \theta, \quad \dot{\omega}_z = 0
\]

Applying

\[
\Sigma M_x = I_x \dot{\omega}_x + (I_z - I_y)\omega_x \omega_y
\]

\[
Fr \sin \theta - N r \cos \theta = \frac{m r^2}{2} (-\dot{\phi} \psi \sin \theta) + (m r^2 - \frac{mr^2}{2})(-\dot{\psi} + \dot{\phi} \cos \theta)(\dot{\phi} \sin \theta)
\]

Using Eqs. (1), (2) and (3), and eliminating \( \dot{\theta} \), we have

\[
m a \dot{\phi}^2 r \sin \theta - m g r \cos \theta = \frac{m r^2}{2} (-\dot{\phi} \psi \sin \theta) + \frac{mr^2}{2} (-\dot{\phi} a \cos \theta)
\]

\[
m a \dot{\phi}^2 \sin \theta r - m g r \cos \theta = \frac{m r^2}{2} (-\dot{\phi}^2 a \cos \theta) + \frac{mr^2}{2} (-\dot{\phi} \psi \sin \theta \cos \theta)
\]

\[
2 g \cos \theta = a \dot{\phi}^2 \sin \theta + r \dot{\phi} \psi \sin \theta \cos \theta
\]

\[
\dot{\phi} = \left( \frac{2g \cot \theta}{a + r \cos \theta} \right)^{1/2}
\]  \hspace{1cm} Ans.

\[
\phi = \left( \frac{2g \cos \theta}{a + r \cos \theta} \right)^{1/2}
\]  \hspace{1cm} Ans.
The conical top has a mass of 0.8 kg, and the moments of inertia are $I_x = I_y = 3.5 \times 10^{-3}$ kg m$^2$ and $I_z = 0.8 \times 10^{-3}$ kg m$^2$. If it spins freely in the ball-and-socket joint at $A$ with an angular velocity $\omega_s = 750$ rad/s, compute the precession of the top about the axis of the shaft $AB$.

**SOLUTION**

$\omega_s = 750$ rad/s

Using Eq. 21-30.

$$\Sigma M_x = -I \dot{\phi}^2 \sin \theta \cos \theta + I_z \dot{\phi} \sin \theta (\dot{\phi} \cos \theta + \dot{\phi})$$

$$0.1(0.8)(9.81) \sin 30^\circ = -3.5 \times 10^{-3} \dot{\phi}^2 \sin 30^\circ \cos 30^\circ + 0.8 \times 10^{-3} \dot{\phi} \sin 30^\circ (\dot{\phi} \cos 30^\circ + 750)$$

Thus,

$$1.160 \times 10^{-3} \dot{\phi}^2 - 300 \times 10^{-3} \dot{\phi} + 0.3924 = 0$$

$\dot{\phi} = 1.31$ rad/s (low precession)  \hspace{1cm} \text{Ans.}$

$\dot{\phi} = 255$ rad/s (high precession)  \hspace{1cm} \text{Ans.}$
21–69.

The top has a mass of 90 g, a center of mass at \( G \), and a radius of gyration \( k = 18 \text{ mm} \) about its axis of symmetry. About any transverse axis acting through point \( O \) the radius of gyration is \( k_i = 35 \text{ mm} \). If the top is connected to a ball-and-socket joint at \( O \) and the precession is \( \omega_p = 0.5 \text{ rad/s} \), determine the spin \( \omega_s \).

**SOLUTION**

\[
\omega_p = 0.5 \text{ rad/s}
\]

\[
\Sigma M_x = -I \dot{\phi}^2 \sin \theta \cos \theta + I \phi \sin \theta \left( \phi \cos \theta + \psi \right)
\]

\[
0.090(9.81)(0.06) \sin 45^\circ = -0.090(0.035)^2 \left( 0.5 \right)^2 \left( 0.7071 \right)^2 + 0.090(0.018)^2(0.5)(0.7071) \left[ 0.5(0.7071) + \psi \right]
\]

\[
\omega_s = \psi = 3.63 \left( 10^3 \right) \text{ rad/s}
\]

**Ans:**

\[
\omega_s = 3.63 \left( 10^3 \right) \text{ rad/s}
\]
21–70.

The 1-lb top has a center of gravity at point $G$. If it spins about its axis of symmetry and precesses about the vertical axis at constant rates of $\omega_x = 60 \text{ rad/s}$ and $\omega_p = -10 \text{ rad/s}$, respectively, determine the steady state angle $\theta$. The radius of gyration of the top about the $z$ axis is $k_z = 1 \text{ in.}$, and about the $x$ and $y$ axes it is $k_x = k_y = 4 \text{ in.}$

**SOLUTION**

Since $\dot{\psi} = \omega_x = 60 \text{ rad/s}$ and $\dot{\phi} = \omega_p = -10 \text{ rad/s}$ and $\theta$ are constant, the top undergoes steady precession.

$$I_z = \left(\frac{1}{32.2}\right)\left(\frac{1}{12}\right)^2 = 215.67\left(10^{-6}\right) \text{ slug} \cdot \text{ft}^2 \quad \text{and} \quad I = I_x = I_y = \left(\frac{1}{32.2}\right)\left(\frac{4}{12}\right)^2 = 3.4507\left(10^{-3}\right) \text{ slug} \cdot \text{ft}^2.$$  

Thus,

$$\Sigma M_x = -I\dot{\phi}^2 \sin \theta \cos \theta + I_z \dot{\phi} \sin \theta (\phi \cos \dot{\psi} + \dot{\phi})$$

$$-1 \sin \theta(0.25) = -3.4507\left(10^{-3}\right)(-10)^2 \sin \theta \cos \theta + 215.67\left(10^{-6}\right)(-10) \sin \theta (-10 \cos \theta + 60)$$

$$\theta = 68.1^\circ \quad \text{Ans.}$$
21–71.

The space capsule has a mass of 2 Mg, center of mass at \( G \), and radii of gyration about its axis of symmetry (\( z \) axis) and its transverse axes (\( x \) or \( y \) axis) of \( k_z = 2.75 \text{ m} \) and \( k_x = k_y = 5.5 \text{ m} \), respectively. If the capsule has the angular velocity shown, determine its precession \( \dot{\phi} \) and spin \( \dot{\psi} \). Indicate whether the precession is regular or retrograde. Also, draw the space cone and body cone for the motion.

**SOLUTION**

The only force acting on the space capsule is its own weight. Thus, it undergoes torque-free motion. \( I_z = 2000(2.75^2) = 15\,125 \text{ kg m}^2 \), \( I = I_x = I_y = 2000(5.5^2) = 60\,500 \text{ kg m}^2 \). Thus,

\[
\omega_y = \frac{H_G \sin \theta}{I},
\]

\[
150 \sin 30^\circ = \frac{H_G \sin \theta}{60\,500}
\]

\[
H_G \sin \theta = 4\,537\,500 \tag{1}
\]

\[
\omega_z = \frac{H_G \cos \theta}{I_z},
\]

\[
150 \cos 30^\circ = \frac{H_G \cos \theta}{15\,125}
\]

\[
H_G \cos \theta = 1\,964\,795.13 \tag{2}
\]

Solving Eqs. (1) and (2),

\[
H_G = 4.9446\,(10^6) \text{ kg m}^2/\text{s} \quad \theta = 66.59^\circ
\]

Using these results,

\[
\dot{\phi} = \frac{H_G}{I} = \frac{H_G}{60\,500} = \frac{4.9446\,(10^6)}{60\,500} = 81.7 \text{ rad/s} \quad \text{Ans.}
\]

\[
\dot{\psi} = \frac{I - I_z}{2I_z} H_G \cos \theta = \left[ \frac{60\,500 - 15\,125}{60\,500(15\,125)} \right] 4.9446\,(10^6) \cos 30^\circ
\]

\[
= 212 \text{ rad/s} \quad \text{Ans.}
\]

Since \( I > I_z \), the motion is regular precession. 

**Ans:**

\[
\dot{\phi} = 81.7 \text{ rad/s}
\]

\[
\dot{\psi} = 212 \text{ rad/s}
\]

regular precession
*21–72. The 0.25 kg football is spinning at \( \omega_z = 15 \text{ rad/s} \) as shown. If \( \theta = 40^\circ \), determine the precession about the \( z \) axis. The radius of gyration about the spin axis is \( k_z = 0.042 \text{ m} \), and about a transverse axis is \( k_y = 0.13 \text{ m} \).

**SOLUTION**

Here, \( \dot{\phi} = \omega_r = 15 \text{ rad/s} \), \( I = mk_z^2 = 0.25(0.13^2) = 0.004225 \text{ kg} \cdot \text{m}^2 \) and \( I_z = mk_y^2 = 0.25(0.042^2) = 0.000441 \text{ kg} \cdot \text{m}^2 \).

\[
\dot{\phi} = \frac{I - I_z}{I_z} \phi \cos \theta; \quad 15 = \left( \frac{0.004225 - 0.000441}{0.000441} \right) \phi \cos 40^\circ
\]

\[
\phi = 2.282 \text{ rad/s}^2 = 2.28 \text{ rad/s}^2 \quad \text{Ans.}
\]
21–73.

The projectile shown is subjected to torque-free motion. The transverse and axial moments of inertia are \( I \) and \( I_z \), respectively. If \( \theta \) represents the angle between the precessional axis \( Z \) and the axis of symmetry \( z \), and \( \beta \) is the angle between the angular velocity \( \omega \) and the \( z \) axis, show that \( \beta \) and \( \theta \) are related by the equation \( \tan \theta = \left( I/I_z \right) \tan \beta \).

**SOLUTION**

From Eq. 21–34 \( \omega_y = \frac{H_G \sin \theta}{I} \) and \( \omega_z = \frac{H_G \cos \theta}{I_z} \) Hence \( \frac{\omega_y}{\omega_z} = \frac{I_z}{I} \tan \theta \)

However, \( \omega_y = \omega \sin \beta \) and \( \omega_z = \omega \cos \beta \)

\[
\frac{\omega_y}{\omega_z} = \tan \beta = \frac{I_z}{I} \tan \theta
\]

\[
\tan \theta = \frac{I}{I_z} \tan \beta \quad \text{Q.E.D.}
\]
SOLUTION

\[ I = 1600(1.8)^2, \quad I_z = 1600(1.2)^2 \]

Use the result of Prob. 21–75.

\[
\tan \theta = \left( \frac{I}{I_z} \right) \tan \beta \\
\tan 20^\circ = \left( \frac{1600(1.8)^2}{1600(1.2)^2} \right) \tan \beta \\
\beta = 9.189^\circ
\]

Using the law of sines:

\[
\frac{\sin 9.189^\circ}{2} = \frac{\sin (20^\circ - 9.189^\circ)}{\psi}
\]

\[ \dot{\psi} = 2.35 \text{ rev/h} \quad \text{Ans.} \]
21–75.

The rocket has a mass of 4 Mg and radii of gyration $k_x = 0.85$ m and $k_y = 2.3$ m. It is initially spinning about the $z$ axis at $\omega_z = 0.05$ rad/s when a meteoroid $M$ strikes it at $A$ and creates an impulse $I = \{300\} \text{ N} \cdot \text{s}$. Determine the axis of precession after the impact.

SOLUTION

The impulse creates an angular momentum about the $y$ axis of

$$H_y = 300(3) = 900 \text{ kg} \cdot \text{m}^2/\text{s}$$

Since

$$\omega_z = 0.05 \text{ rad/s}$$

then

$$H_G = 900\mathbf{j} + [4000(0.85)^2](0.05)\mathbf{k} = 900\mathbf{j} + 144.5\mathbf{k}$$

The axis of precession is defined by $H_G$

$$u_{H_G} = \frac{900\mathbf{j} + 144.5\mathbf{k}}{911.53} = 0.9874\mathbf{j} + 0.159\mathbf{k}$$

Thus,

$$\alpha = \cos^{-1}(0) = 90^\circ$$

$$\beta = \cos^{-1}(0.9874) = 9.12^\circ$$

$$\gamma = \cos^{-1}(0.159) = 80.9^\circ$$

Ans: $\alpha = 90^\circ$, $\beta = 9.12^\circ$, $\gamma = 80.9^\circ$
The football has a mass of 450 g and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of \( k_z = 30\text{mm} \) and \( k_x = k_y = 50\text{mm} \), respectively. If the football has an angular momentum of \( H_G = 0.02\text{kg} \cdot \text{m}^2/\text{s} \), determine its precession \( \phi \) and spin \( \psi \). Also, find the angle \( \beta \) that the angular velocity vector makes with the z axis.

**SOLUTION**

Since the weight is the only force acting on the football, it undergoes torque-free rotation. \( I_z = 0.45(0.03^2) = 0.405(10^{-3}) \text{ kg} \cdot \text{m}^2 \), \( I = I_z = I_y = 0.45(0.05^2) = 1.125(10^{-3}) \text{ kg} \cdot \text{m}^2 \), and \( \theta = 45^\circ \).

Thus,

\[
\dot{\phi} = \frac{H_G}{I} = \frac{0.02}{1.125(10^{-3})} = 17.8 \text{ rad/s} = 17.8 \text{ rad/s}
\]

\[\text{Ans.}\]

\[
\dot{\psi} = \frac{I - I_z}{2I_z} H_G \cos \theta = \frac{1.125(10^{-3})}{1.125(10^{-3})(0.405)(10^{-3})}(0.02) \cos 45^\circ \\
= 22.35 \text{ rad/s} = 22.3 \text{ rad/s}
\]

\[\text{Ans.}\]

Also,

\[
\omega_y = \frac{H_G \sin \theta}{I} = \frac{0.02 \sin 45^\circ}{1.125(10^{-3})} = 12.57 \text{ rad/s}
\]

\[
\omega_z = \frac{H_G \cos \theta}{I_z} = \frac{0.02 \cos 45^\circ}{0.405(10^{-3})} = 34.92 \text{ rad/s}
\]

Thus,

\[
\beta = \tan^{-1}\left(\frac{\omega_y}{\omega_z}\right) = \tan^{-1}\left(\frac{12.57}{34.92}\right) = 19.8^\circ
\]

\[\text{Ans.}\]
21–77.

The satellite has a mass of 1.8 Mg, and about axes passing through the mass center $G$ the axial and transverse radii of gyration are $k_z = 0.8 \text{ m}$ and $k_t = 1.2 \text{ m}$, respectively. If it is spinning at $\omega_s = 6 \text{ rad/s}$ when it is launched, determine its angular momentum. Precession occurs about the $Z$ axis.

**SOLUTION**

\[
I = 1800(1.2)^2 = 2592 \text{ kg} \cdot \text{m}^2 \quad I_z = 1800(0.8)^2 = 1152 \text{ kg} \cdot \text{m}^2
\]

Applying the third of Eqs. 21–36 with $\theta = 5^\circ$ $\psi = 6 \text{ rad/s}$

\[
\psi = \frac{I - I_z}{H_z} H_G \cos \theta
\]

\[
0 = \frac{2592 - 1152}{2592(1152)} H_G \cos 5^\circ
\]

\[
H_G = 12.5 \text{ Mg} \cdot \text{m}^2/\text{s}
\]

Ans.
21–78.

The radius of gyration about an axis passing through the axis of symmetry of the 1.2 Mg satellite is \( k_z = 1.4 \text{ m} \), and about any transverse axis passing through the center of mass \( G \), \( k_t = 2.20 \text{ m} \). If the satellite has a known spin of 2700 rev/h about the \( z \) axis, determine the steady-state precession about the \( z \) axis.

**SOLUTION**

*Gyroscopic Motion:* Here, the spinning angular velocity \( \omega_s = \frac{2700(2\pi)}{3600} = 1.5\pi \text{ rad/s} \).

The moment inertia of the satellite about the \( z \) axis is \( I_z = 1200\left(1.4^2\right) = 2352 \text{ kg} \cdot \text{m}^2 \) and the moment inertia of the satellite about its transverse axis is \( I = 1200\left(2.20^2\right) = 5808 \text{ kg} \cdot \text{m}^2 \).

Applying the third of Eq. 21–36 with \( \theta = 15^\circ \), we have

\[
\psi = \frac{I - I_z}{I I_z} H_G \cos \theta
\]

\[
1.5\pi = \left[ \frac{5808 - 2352}{5808\left(2352\right)} \right] H_G \cos 15^\circ
\]

\[
H_G = 19.28\left(10^3\right) \text{ kg} \cdot \text{m}^2/\text{s}
\]

Applying the second of Eq. 21–36, we have

\[
\dot{\phi} = \frac{H_G}{I} = \frac{19.28\left(10^3\right)}{5808} = 3.32 \text{ rad/s}
\]

Ans.

\[
\dot{\phi} = 3.32 \text{ rad/s}
\]